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Department of Mathematics

Preparation Guide

for

NGAT Examination

Quantitative Reasoning

Study Material for Master's and PhD Applicants

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Contents

1 Percentage, Ratio, Age Problems

1.1 Percentage: The Language of 100

A percentage is simply a specific type of fraction where the denominator is always 100. It allows us to compare "parts" of "wholes" even when the wholes are different sizes. Most NGAT problems ask for one of three things: the Part, the Whole, or the Rate.

- **Finding the Part:** Whole $\times$ Percentage (in decimal form)
- **Finding the Whole:** $\frac{\text{Part}}{\text{Percentage (in decimal form)}}$
- **Percentage Change:** $\frac{\text{New Value}-\text{Old Value}}{\text{Old Value}} \times 100$

Strategic Tip: The "100" Substitution

If a percentage problem does not provide a starting value (e.g., "A price increases by 20% and then decreases by 10%"), **assume the starting value is 100**.

1. $100 + 20\% = 120$
2. $120 - 10\%(120) = 108$
3. Net change $= 8\%$ increase.

1.2 Ratio: Proportional Comparison

A ratio $(a : b)$ tells you how many times one number contains another.

NGAT ratio problems are best solved by assigning a variable x to the "common multiplier." If the ratio of boys to girls is $3 : 5$, let:

$$\textbf{Boys} = 3x \quad \textbf{and} \quad \textbf{Girls} = 5x$$

To divide a total amount (T) into a ratio $a : b$:

1. Find the **Total Parts**: $a + b$
2. Find the **Value of One Part**: $T/(a + b)$
3. **Distribute**: Multiply the one-part value by a and b.

1.3 Age Problems

Age problems are essentially logic puzzles that require translating sentences into linear equations.

1.4 The Two Golden Rules of Age

1. **Constant Difference:** If Jack is 10 years older than Jill today, he will be 10 years older in 50 years. *Subtraction between ages always yields the same constant.*
2. **Moving in Time:** When you move n years into the future or past, you must add/subtract n from **every** person mentioned in the problem.

NGAT Strategy: The Age Table

For complex problems involving past and future, always draw a table:

Person	**Past** $(x-5)$	**Present** (x)	**Future** $(x+10)$
Father	$F-5$	F	$F+10$
Son	$S-5$	S	$S+10$

1.5 Quick Tips & Reality Checks

- **The 10% Shortcut:** To find 10% of a number, move the decimal one place left. To find 5%, find 10% and cut it in half.
- **Unit Consistency:** In ratios, never compare 2 kg to 500 g directly. Convert to 2000 g : 500 g (4 : 1) first.
- **Sanity Check:** In age problems, if you get a negative number or a child older than a parent, check if you accidentally subtracted years instead of adding them for a "future" statement.

1.6 Examples

1. If $\frac{1}{3}$ of $\frac{1}{4}$ of a number is 15, then $\frac{3}{10}$ of that number is:

   A. 35

   B. 36

   C. 45

   D. 54

   Solution:
   Let the unknown number be denoted by x.

   According to the given condition, one-third of one-fourth of this number is equal to 15. We can translate this statement into the following algebraic equation:

$$\frac{1}{3} \times \frac{1}{4} \times x = 15$$

Simplifying the product of the fractions on the left-hand side yields:

$$\frac{1}{12}x = 15$$

To isolate x, we multiply both sides of the equation by 12:

$$x = 15 \times 12$$
$$x = 180$$

Now that we have determined the number is 180, we find three-tenths of this number as required by the problem:

$$\begin{aligned}\text{Required Value} &= \frac{3}{10} \times x \\ &= \frac{3}{10} \times 180 \\ &= 3 \times 18 \\ &= 54\end{aligned}$$

Thus, three-tenths of the number is 54, which corresponds to option **D**.

2. Three times the first of three consecutive odd integers is 3 more than twice the third. The third integer is:

   A. 9

   B. 11

   C. 13

   D. 15

   Solution:
   Let the three consecutive odd integers be denoted as follows:

$$\begin{aligned}\text{First integer} &= n \\ \text{Second integer} &= n + 2 \\ \text{Third integer} &= n + 4\end{aligned}$$

   where n is an odd integer.

   According to the problem statement, "three times the first integer" is equal to "3 more than twice the third integer". We can translate this condition into the following algebraic equation:

$$3n = 2(n + 4) + 3$$

   Now, we solve for n by first expanding the right-hand side of the equation:

$$3n = 2n + 8 + 3$$

   Combining the constant terms yields:

$$3n = 2n + 11$$

Subtracting $2n$ from both sides of the equation isolates the variable n:

$$n = 11$$

Since the problem specifically asks for the value of the *third* integer, we substitute $n = 11$ back into our expression for the third integer:

$$\begin{aligned}\text{Third integer} &= n + 4 \\ &= 11 + 4 \\ &= 15\end{aligned}$$

Thus, the third integer is 15, which corresponds to option **D**.

1.7 Self-test Exercise

1. A student's score increased from 60 to 75. What was the percentage increase?

(A) 15%
(B) 20%
(C) 25%
(D) 30%

2. The ratio of two numbers is $4 : 7$. If their sum is 132, what is the smaller number?

(A) 48
(B) 52
(C) 84
(D) 33

3. Five years ago, a father was 3 times as old as his son. If the sum of their current ages is 50, how old is the father now?

(A) 30
(B) 35
(C) 40
(D) 45

Answers and Explanations

1. Correct Answer: (C) 25%

Explanation:

$$\text{Percentage Increase} = \frac{\text{New Value} - \text{Original Value}}{\text{Original Value}} \times 100 = \frac{75 - 60}{60} \times 100 = \frac{15}{60} \times 100 = 25\%.$$

2. **Correct Answer: (A) 48**

   Explanation:

   Let the common factor be x. Then the two numbers are $4x$ and $7x$.

$$4x + 7x = 132$$

$$11x = 132$$

$$x = 12.$$

   Hence, the smaller number is

$$4 \times 12 = 48.$$

3. **Correct Answer: (B) 35**

   Explanation:

   Let the father's current age be F and the son's current age be S.

$$F + S = 50.$$

   Five years ago,

$$F - 5 = 3(S - 5).$$

   Expanding,

$$F - 5 = 3S - 15,$$

$$F = 3S - 10.$$

   Substitute into the first equation:

$$3S - 10 + S = 50,$$

$$4S = 60,$$

$$S = 15.$$

   Therefore,

$$F = 50 - 15 = 35.$$

2 Logic and Set Theory

2.1 Propositional Logic: Decoding Analytical Reasoning

The NGAT tests your ability to evaluate the validity of arguments, independent of real-world biases.
A **proposition** is simply a declarative statement that is either strictly True (T) or strictly False (F).

2.2 Logical Connectives

NGAT analytical questions often hide logical operators in standard English prose. Use this translation guide:

- **Negation ($\neg p$):** "Not p," "It is false that p." It completely reverses the truth value.
- **Conjunction ($p \wedge q$):** "p **and** q." True *only* if both individual statements are true.
- **Disjunction ($p \vee q$):** "p **or** q," "At least one of them." True if either one or both are true.
- **Conditional ($p \implies q$):** "**If** p, **then** q," "q, **whenever** p," "p **is sufficient for** q," "q **is necessary for** p."

Conditional statement

The statement $p \implies q$ is a one-way implication! It is only **False** when a True premise (p) leads to a False conclusion (q). If p does not happen (F), the overall statement remains automatically **True**, regardless of q.

When the NGAT presents a conditional statement, they will often ask you to identify a logically equivalent statement. Given the original statement **"If it rains (p), then the ground is wet (q)"**:

1. **Converse ($q \implies p$):** "If the ground is wet, then it rained." *(Not logically equivalent! The ground could be wet from a sprinkler.)*
2. **Inverse ($\neg p \implies \neg q$):** "If it does not rain, then the ground is not wet." *(Not logically equivalent!)*
3. **Contrapositive ($\neg q \implies \neg p$):** "If the ground is not wet, then it did not rain." **(Always Logically Equivalent to the Original!)**

2.3 Unified Truth Table Lookup

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \implies q$ (Original)	$\neg q \implies \neg p$ (Contrapositive)
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	T
F	F	T	F	F	T	T

2.4 Set Theory: Solving Categorical & Counting Problems

NGAT quantitative sections frequently use sets to test overlapping classifications (e.g., students taking multiple courses, consumers buying distinct brands).

2.5 Visualizing Set Operations

Instead of memorizing symbols, visualize them as regions of a Venn diagram:

- **Union** ($A \cup B$)**:** "A **or** B." The entire combined area of both circles.
- **Intersection** ($A \cap B$)**:** "Both A **and** B." The central overlapping football shape.
- **Complement** (A' **or** A^c)**:** "Not A." Everything outside circle A.
- **Difference** ($A \setminus B$)**:** "A only." The crescent moon shape left behind when you cut circle B out of circle A.

2.6 The NGAT Counting Blueprint (Inclusion-Exclusion)

When word problems provide numbers for overlapping groups, direct addition results in double-counting the candidates who belong to both. Use these strict formulas:

For Two Overlapping Categories:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For Three Overlapping Categories:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Step-by-Step NGAT Problem Solving Strategy

When tackling a 2 or 3-circle Venn diagram word problem:

1. **Start from the center:** Always plug in the value for the innermost intersection first ($A \cap B \cap C$ or $A \cap B$).
2. **Work your way outwards:** Subtract this inner value from the double-intersection values before writing them into the outer sectors.
3. **Account for the outsiders:** Don't forget that the Universal Set (U) often includes elements that belong to *none* of the given sets.

2.6.1 Strategic Shortcuts for NGAT Success

- **Elimination via Contradiction:** In the analytical reasoning section, if an answer choice contradicts a known truth-table rule (such as assuming the Converse is true), eliminate it immediately.
- **De Morgan's Quick Check:** If a question asks you to negate a joint option like "You must pass the exam **and** pay the fee," the correct negation flips both terms and switches the operator: "You do **not** pass the exam **or** you do **not** pay the fee."
- **Sanity Check on Totals:** In set counting problems, ensure that your calculated total ($A \cup B$) never exceeds the total population size given in the prompt.

2.7 Examples

Solved Problem 1: Contrapositive Translation

Problem. NGAT analytical reasoning prompt states: "If an applicant does not pass the English proficiency module, they are not admitted to the postgraduate program." Which of the following statements is logically equivalent?

(A) If an applicant passes the English proficiency module, they are admitted.

(B) If an applicant is admitted to the postgraduate program, they passed the English proficiency module.

(C) If an applicant is not admitted to the postgraduate program, they did not pass the English proficiency module.

(D) An applicant is admitted if and only if they pass the English proficiency module.

Solution:
Let p be the statement: "The applicant does not pass the English proficiency module."
Let q be the statement: "The applicant is not admitted."
The original statement is in the conditional form: $p \implies q$.
According to the rules of propositional logic, a conditional statement is always logically equivalent to its **contrapositive**: $\neg q \implies \neg p$.

- $\neg q$: Negation of "not admitted" $\to$ "The applicant is admitted."
- $\neg p$: Negation of "does not pass" $\to$ "The applicant passed the English proficiency module."

Constructing $\neg q \implies \neg p$ gives: "If an applicant is admitted to the postgraduate program, they passed the English proficiency module."
Answer:
$\boxed{\text{B}}$

Solved Problem 2: Two-Category Inclusion-Exclusion

Problem. In a group of 120 graduate candidates sitting for the NGAT exam, 75 passed the Quantitative Reasoning section and 65 passed the Verbal Reasoning section. If 30 candidates passed both sections, how many candidates failed both sections?

Solution:
Let Q be the set of candidates who passed Quantitative Reasoning, and V be the set of candidates who passed Verbal Reasoning. We are given the following cardinalities:

$$|U| = 120, \quad |Q| = 75, \quad |V| = 65, \quad |Q \cap V| = 30$$

Using the Two-Category Inclusion-Exclusion formula to find the total number of candidates who passed at least one section ($|Q \cup V|$):

$$|Q \cup V| = |Q| + |V| - |Q \cap V|$$

$$|Q \cup V| = 75 + 65 - 30 = 110$$

The number of candidates who failed both sections is the complement of the union relative to the universal set U:

$$\text{Failed Both} = |U| - |Q \cup V| = 120 - 110 = 10$$

Answer:
$\boxed{\text{10 candidates}}$

Solved Problem 3: De Morgan's Law in Logic

Problem. Find the exact logical negation of the structural rule: "A candidate must achieve a score above 600 **and** submit two letters of recommendation."

Solution:
Let p be: "A candidate achieves a score above 600."
Let q be: "A candidate submits two letters of recommendation."
The rule is a conjunction: $p \wedge q$.
To find the negation, apply **De Morgan's Laws**: $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

- $\neg p$: "A candidate does not achieve a score above 600."
- $\neg q$: "A candidate does not submit two letters of recommendation."
- The connective switches from **and** ($\wedge$) to **or** ($\vee$).

Combining these yields: "A candidate does not achieve a score above 600 **or** does not submit two letters of recommendation."

Answer:
A candidate does not achieve a score above 600 OR does not submit two letters of recommendatio

Solved Problem 4: Set Difference and 'Only' Word Problems

Problem. Out of 80 agricultural officers surveyed regarding epidemic mitigation, 45 prefer using digital notification systems (A), 35 prefer using local community networks (B), and 15 use both methods simultaneously. How many officers prefer using digital notification systems **only**?

Solution:
We are looking for the cardinality of the set difference $A \setminus B$, which represents elements belonging exclusively to A after removing the shared intersection.

$$|A| = 45, \quad |B| = 35, \quad |A \cap B| = 15$$

Using the set difference formula:

$$|A \setminus B| = |A| - |A \cap B|$$

$$|A \setminus B| = 45 - 15 = 30$$

Answer:
$\boxed{\text{30 officers}}$

Solved Problem 5: Evaluating Conditional Truth Values

Problem. Suppose the statement "If the data collection is incomplete (p), then the simulation model crashes (q)" is evaluated as mathematically **True**. If it is discovered that the simulation model did **not** crash, what can be definitively concluded about the completeness of the data collection?

Solution:
We are given that the conditional statement $p \implies q$ is T, and we know that q is F (the model did not crash).
Let us inspect the Unified Truth Table for $p \implies q$:

- If $p = T$ and $q = F$, then $p \implies q = F$ (Contradicts our given information).
- If $p = F$ and $q = F$, then $p \implies q = T$ (Matches our given information).

Therefore, p must be strictly **False**. Since p represents "the data collection is incomplete," its falsehood implies that the data collection is complete.
Answer:
$\boxed{\text{The data collection must be complete.}}$

Solved Problem 6: Three-Category Center-Outwards Venn Diagram

Problem. An evaluation committee tests 100 industrial software scripts for security parameters: Encryption (E), Authentication (A), and Redundancy (R).

- 48 scripts use Encryption (E)
- 39 scripts use Authentication (A)
- 30 scripts use Redundancy (R)
- 14 use both E and A
- 12 use both E and R
- 10 use both A and R
- 4 scripts implement all three safety components ($E \cap A \cap R$)

How many scripts implement **none** of these parameters?

Solution:
Apply the strict three-category Inclusion-Exclusion blueprint to evaluate the unified core $|E \cup A \cup R|$:

$$|E \cup A \cup R| = |E| + |A| + |R| - |E \cap A| - |E \cap R| - |A \cap R| + |E \cap A \cap R|$$

Substitute the given values into the identity:

$$|E \cup A \cup R| = 48 + 39 + 30 - 14 - 12 - 10 + 4$$

$$|E \cup A \cup R| = 117 - 36 + 4 = 85$$

The number of scripts implementing none of these security features is the universal complement:

$$\text{None} = |U| - |E \cup A \cup R| = 100 - 85 = 15$$

Answer:
$\boxed{\text{15 scripts}}$

Solved Problem 7: Invalid Argument Forms (Converse Error)

Problem. Consider the premise: "Whenever a country encounters capital stagnation, its market currency devalues." Assuming this statement is true, an analyst observes that the country's market currency has just devalued. The analyst concludes: "Therefore, the country is experiencing capital stagnation." Is this argument valid?

Solution:
Let p: "The country encounters capital stagnation."
Let q: "Its market currency devalues."
The underlying logic statement is a conditional form: $p \implies q$.
The analyst observes that q occurs and concludes that p must follow. This sets up the logical form: $q \implies p$. As established in our core text, the Converse ($q \implies p$) is **not logically equivalent** to the original conditional expression. Currency devaluation could be driven by distinct external variables (such as central bank policy modifications or sudden import shifts) without requiring capital stagnation.
Answer:
No, this is an invalid argument known as the Converse Error.

Solved Problem 8: Finding Regions via Venn Subtraction

Problem. In a medical review of 60 chronic tuberculosis patients, 34 are on Treatment Regime Alpha (A) and 26 are on Treatment Regime Beta (B). If 12 patients are on neither regime, how many patients are on Regime Beta **only**?

Solution:
First, determine the total number of patients who are on at least one regime ($|A \cup B|$):

$$|A \cup B| = |U| - \text{Neither} = 60 - 12 = 48$$

Now, use the Inclusion-Exclusion formula to isolate the shared overlap intersection ($|A \cap B|$):

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$48 = 34 + 26 - |A \cap B|$$

$$48 = 60 - |A \cap B| \implies |A \cap B| = 12$$

To find the number of patients on Regime Beta **only** ($|B \setminus A|$), subtract the overlap from the total population of B:

$$|B \setminus A| = |B| - |A \cap B| = 26 - 12 = 14$$

Answer:
$\boxed{\text{14 patients}}$

Solved Problem 9: Logical Implication of False Premise

Problem. Determine the structural truth value of the following conditional mathematical proposition: "If $3 + 2 = 8$, then $\lim_{x \to 0} \frac{\sin x}{x} = 5$."

Solution:
Let p be the premise: "$3 + 2 = 8$." This statement is clearly **False** (F).
Let q be the conclusion: "$\lim_{x \to 0} \frac{\sin x}{x} = 5$." This statement is also **False** (F).
According to the rule of conditional statements ($p \implies q$), a conditional relation is evaluated as strictly **True** whenever the initial premise is False. Because a false statement cannot logically lead to a violation of a one-way implication, the overall implication remains vacuously valid under truth-table rules.
Answer:
$\boxed{\text{True (Vacuously True)}}$

Solved Problem 10: Three-Set Missing Intersection

Problem. A local university tracks 100 students majoring in computing sciences. 50 students take Python (P), 40 take R (R), and 30 take MATLAB (M). The records show that 15 take both Python and R, 10 take Python and MATLAB, and 8 take R and MATLAB. If 5 students are registered for all three languages, how many students are taking **at least one** of these three languages?

Solution:
We are directly solving for the cardinality of the union of three overlapping sets, defined as $|P \cup R \cup M|$. We use our step-by-step numbers in the explicit three-set identity:

- ✓ Individual Totals: $|P| = 50, \quad |R| = 40, \quad |M| = 30$
- ✓ Double Intersections: $|P \cap R| = 15, \quad |P \cap M| = 10, \quad |R \cap M| = 8$
- ✓ Triple Intersection: $|P \cap R \cap M| = 5$

Substitute the complete parameters directly into the identity:

$$|P \cup R \cup M| = (|P| + |R| + |M|) - (|P \cap R| + |P \cap M| + |R \cap M|) + |P \cap R \cap M|$$

$$|P \cup R \cup M| = (50 + 40 + 30) - (15 + 10 + 8) + 5$$

$$|P \cup R \cup M| = 120 - 33 + 5 = 92$$

This confirms that 92 out of the 100 students are enrolled in at least one of the language modules.
Answer:
$\boxed{\text{92 students}}$

2.8 Self test exercise

1. Let p represent the statement "The student passes the examination." Which of the following represents the negation of p?

   A. The student passes the examination and graduates.

   B. The student does not pass the examination.

   C. The student graduates.

   D. The student may pass the examination.

2. Which of the following statements represents the conjunction of p and q?

   A. p or q

   B. If p, then q

   C. p and q

   D. Not p

3. Let p be "It is raining" and q be "The road is wet." Which statement is the converse of $p \implies q$?

   A. If the road is wet, then it is raining.

   B. If it is not raining, then the road is not wet.

   C. If the road is not wet, then it is not raining.

   D. It is raining and the road is wet.

4. Which statement is logically equivalent to the conditional statement $p \implies q$?

   A. $q \implies p$

   B. $\neg p \implies \neg q$

   C. $\neg q \implies \neg p$

   D. $p \wedge q$

5. If p is False and q is True, what is the truth value of $p \implies q$?

   A. True

   B. False

   C. Cannot be determined

   D. Both true and false

6. The negation of the statement "Ali passed the exam and paid the fee" is:

   A. Ali did not pass the exam and did not pay the fee.

   B. Ali passed the exam or paid the fee.

   C. Ali did not pass the exam or did not pay the fee.

D. Ali failed the exam and paid the fee.

7. In a class of 50 students, 28 study Biology, 22 study Chemistry, and 10 study both subjects. How many students study at least one of the two subjects?

   A. 40
   B. 50
   C. 60
   D. 70

8. In a survey, 45 people drink tea, 35 drink coffee, and 15 drink both. How many people drink either tea or coffee?

   A. 65
   B. 70
   C. 75
   D. 80

9. A group of 100 students was surveyed. Sixty students play football, 50 play basketball, and 20 play both. How many students play neither sport?

   A. 5
   B. 10
   C. 15
   D. 20

10. Let $U = 1, 2, 3, 4, 5, 6, 7, 8$ and $A = 1, 3, 5, 7$. What is the complement of A?

    A. $1, 3, 5, 7$
    B. $2, 4, 6, 8$
    C. $1, 2, 3, 4$
    D. $5, 6, 7, 8$

11. In a university, 80 students take Mathematics, 70 take Physics, and 30 take both courses. How many students take exactly one of the two courses?

    A. 90
    B. 100
    C. 120
    D. 150

12. A survey of 120 households revealed that 65 subscribe to Internet Service A, 55 subscribe to Internet Service B, and 25 subscribe to both. How many subscribe to neither service?

A. 15
B. 20
C. 25
D. 30

13. Which of the following correctly represents $A \setminus B$?

    A. Elements belonging to both A and B
    B. Elements belonging to A but not to B
    C. Elements belonging to B but not to A
    D. All elements outside both sets

14. In a survey of 100 students:

    - 55 study Mathematics,
    - 45 study Physics,
    - 30 study Chemistry,
    - 20 study both Mathematics and Physics,
    - 10 study both Mathematics and Chemistry,
    - 8 study both Physics and Chemistry,
    - 5 study all three subjects.

    How many students study at least one of the three subjects?

    A. 92
    B. 97
    C. 102
    D. 107

15. Which of the following statements is always true?

    A. The converse of a conditional statement.
    B. The inverse of a conditional statement.
    C. The contrapositive of a conditional statement.
    D. The conjunction of two false statements.

Answer Key

1. B 2. C 3. A 4. C 5. A 6. C 7. A 8. A 9. B 10. B 11. A 12. C
13. B 14. B 15. C

3 Relations and functions

3.1 Squares, Square Roots, Exponents, and Logarithms

Short Notes

Squares and Square Roots

Definition: A square of a number is obtained by multiplying the number by itself.
Mathematically it is of the form:

$$a^2 = a \times a$$

Examples of squares of numbers: $3^2 = 9$, $5^2 = 25$, $10^2 = 100$ etc.

Definition: A square root of a number is a value that, when multiplied by itself, gives the original number.
Mathematically it is of the form:

$$\sqrt{a} = b \quad \text{if} \quad b^2 = a$$

Note: Square roots are the inverse of squaring.

Examples of square roots are: $\sqrt{9} = 3$, $\sqrt{25} = 5$, $\sqrt{16} = 4$ etc...

Exponents

Definition: For a natural number n and a real number a, the symbol a^n, read as "the nth power of a" or "a raised to n", is defined as follows:

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ factors}}.$$

In a^n, a is called the **base** and n is called the **exponent**.
Laws of Exponents:

1. $a^m \cdot a^n = a^{m+n}$ ...... law of multiplication of powers of the same base.
2. $(a^m)^n = a^{mn}$...... law of power of a power.
3. $a^0 = 1 \quad (a \neq 0)$
4. $a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$
5. $\frac{a^n}{a^m} = a^{n-m}$..... law of division of powers of the same base.
6. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.... law of a power of a quotient.
7. $a^n b^n = (ab)^n$.... law of a power of a product.

Logarithms

The logarithm to base a of a number $x > 0$ (written $\log_a x$) is the power to which a must be raised to obtain x.
Definition: For $a > 0$, $a \neq 1$, and $c > 0$,

$$\log_a c = b \quad \text{if and only if} \quad a^b = c.$$

Example:

$$\log_2 8 = 3$$

This means that 2 must be raised to the power 3 to obtain 8, i.e.,

$$2^3 = 8.$$

Laws and Properties of Logarithms
Let $a > 0$, $a \neq 1$, and $x, y > 0$.

1. **Product Rule**: $\log_a(xy) = \log_a x + \log_a y$
2. **Quotient Rule**: $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. **Power Rule**: $\log_a(x^r) = r \log_a x$
4. **Logarithm of 1**: $\log_a 1 = 0$
5. **Logarithm of the Base**: $\log_a a = 1$
6. **Inverse Properties**: $\log_a(a^x) = x \quad \text{and} \quad a_{\log_a x} = x$
7. **Change of Base Formula**: $\log_a x = \frac{\log_b x}{\log_b a}, \quad b > 0,\ b \neq 1$
8. **Special cases**: $\log_a x = \frac{\ln x}{\ln a} = \frac{\log x}{\log a}$
9. **Reciprocal Rule**: $\log_a x = \frac{1}{\log_x a}$
10. **Logarithm of a Power of the Base**: $\log_a(a^n) = n$
11. **Equality Property**: If $\log_a x = \log_a y$, then $x = y$.

Solved Questions

Question 1: Application of Exponent Laws

Simplify the following expression and express your answer with positive exponents:

$$\frac{(2x^3y^{-2})^3 \cdot x^2y^5}{4x^5y^{-1}}$$

Solution:

1. First, apply the power of a product and power of a power laws to the numerator term $(2x^3y^{-2})^3$:

$$(2x^3y^{-2})^3 = 2^3 \cdot (x^3)^3 \cdot (y^{-2})^3 = 8x^9y^{-6}$$

2. Substitute this back into the numerator and use the law of multiplication of powers ($a^m \cdot a^n = a^{m+n}$):

$$(8x^9y^{-6}) \cdot (x^2y^5) = 8x^{9+2}y^{-6+5} = 8x^{11}y^{-1}$$

3. Now, combine the simplified numerator with the denominator using the law of division of powers ($\frac{a^n}{a^m} = a^{n-m}$):

$$\frac{8x^{11}y^{-1}}{4x^5y^{-1}} = \left(\frac{8}{4}\right) \cdot x^{11-5} \cdot y^{-1-(-1)}$$
$$= 2 \cdot x^6 \cdot y^0$$

4. Since $y^0 = 1$ (where $y \neq 0$), the final simplified expression is:

$$\mathbf{2x^6}$$

Question 2: Logarithmic Properties and Expansion

Given that $\log_a 2 = 0.301$ and $\log_a 3 = 0.477$, evaluate the exact numerical value of:

$$\log_a\left(\frac{\sqrt{12}}{a^3}\right)$$

Solution:

1. Apply the quotient rule of logarithms ($\log_a(\frac{x}{y}) = \log_a x - \log_a y$):

$$\log_a\left(\frac{\sqrt{12}}{a^3}\right) = \log_a(\sqrt{12}) - \log_a(a^3)$$

2. Convert the radical to a fractional exponent ($\sqrt{12} = 12^{1/2}$) and apply the power rule ($\log_a(x^r) = r\log_a x$):

$$= \frac{1}{2}\log_a(12) - 3\log_a(a)$$

3. Factor 12 into its prime components ($12 = 4 \times 3 = 2^2 \times 3$) and use the logarithm of the base property ($\log_a a = 1$):

$$= \frac{1}{2}\log_a(2^2 \times 3) - 3(1)$$

4. Expand using the product rule ($\log_a(xy) = \log_a x + \log_a y$) and power rule again:

$$= \frac{1}{2}\left[\log_a(2^2) + \log_a 3\right] - 3 = \frac{1}{2}\left[2\log_a 2 + \log_a 3\right] - 3$$

5. Substitute the given values ($\log_a 2 = 0.301$ and $\log_a 3 = 0.477$):

$$= \frac{1}{2}[2(0.301) + 0.477] - 3 = \frac{1}{2}[0.602 + 0.477] - 3$$

$$= \frac{1}{2}(1.079) - 3 = 0.5395 - 3 = -\mathbf{2.4605}$$

Question 3: Solving Exponential Equations Using Logarithms

Solve the following equation for x:

$$5^{2x-1} = 3^{x+2}$$

Solution:

1. Take the common logarithm ($\log_{10}$, written simply as log) of both sides:

$$\log(5^{2x-1}) = \log(3^{x+2})$$

2. Bring the exponents to the front using the power rule:

$$(2x-1)\log 5 = (x+2)\log 3$$

3. Expand both sides of the linear equation:

$$2x\log 5 - \log 5 = x\log 3 + 2\log 3$$

4. Group all terms involving x on the left side and constant terms on the right side:

$$2x\log 5 - x\log 3 = 2\log 3 + \log 5$$

5. Factor out x on the left side:

$$x(2\log 5 - \log 3) = \log(3^2) + \log 5$$

$$x(\log(5^2) - \log 3) = \log 9 + \log 5$$

6. Condense the expressions using the product and quotient rules:

$$x\log\left(\frac{25}{3}\right) = \log(45)$$

7. Isolate x:

$$x = \frac{\log 45}{\log(25/3)} \quad \text{or using Change of Base:} \quad \mathbf{x = \log_{25/3}(45)}$$

Question 4: Solving Logarithmic Equations

Find the solution set for the equation:

$$\log_2(x) + \log_2(x-2) = 3$$

Solution:

1. Combine the logarithms on the left side using the product rule ($\log_a x + \log_a y = \log_a(xy)$):
$$\log_2\big(x(x-2)\big) = 3$$

2. Convert the logarithmic equation into its equivalent exponential form ($a^b = c$):
$$x(x-2) = 2^3$$
$$x^2 - 2x = 8$$

3. Rearrange into a standard quadratic equation form ($ax^2 + bx + c = 0$):
$$x^2 - 2x - 8 = 0$$

4. Factor the quadratic equation to find potential solutions:
$$(x-4)(x+2) = 0$$
This yields two potential solutions: $x = 4$ or $x = -2$.

5. **Check for Extraneous Solutions:** Logarithms are only defined for strictly positive arguments ($x > 0$).
   - For $x = 4$: $\log_2(4)$ and $\log_2(4-2) = \log_2(2)$ are both well-defined.
   - For $x = -2$: $\log_2(-2)$ is undefined because the domain must be strictly greater than 0.

   Therefore, $x = -2$ is an extraneous solution. The true solution set is:
$$\mathbf{x = 4}$$

3.2 Self-test Exercise

1. Evaluate the following:

(A) 11^2

(B) $\sqrt{196}$

(C) $(-8)^2$

(D) $\sqrt{0.64}$

2. Simplify the following expression and express your answer using positive exponents:

$$\frac{(3x^2y^{-1})^2 \cdot x^3y^4}{9xy}.$$

3. Given that $\log_a 2 = 0.301$ and $\log_a 5 = 0.699$, evaluate

$$\log_a\left(\frac{20}{a^2}\right).$$

4. Solve the exponential equation

$$2^{x+1} = 5^{x-1}.$$

5. Solve the logarithmic equation

$$\log_3(x) + \log_3(x-6) = 2.$$

Solutions

1. Solution

(A)

$$11^2 = 11 \times 11 = 121.$$

(B) Since

$$14^2 = 196,$$

we have

$$\sqrt{196} = 14.$$

(C)

$$(-8)^2 = (-8)(-8) = 64.$$

(D) Since

$$0.8^2 = 0.64,$$

therefore,

$$\sqrt{0.64} = 0.8.$$

2. Solution

Apply the power laws:

$$(3x^2y^{-1})^2 = 3^2(x^2)^2(y^{-1})^2 = 9x^4y^{-2}.$$

Multiply by x^3y^4:

$$9x^4y^{-2} \cdot x^3y^4 = 9x^7y^2.$$

Divide by $9xy$:

$$\frac{9x^7y^2}{9xy} = x^{7-1}y^{2-1} = x^6y.$$

Hence,

$$\boxed{x^6y}.$$

3. Solution

Apply the quotient rule:

$$\log_a\left(\frac{20}{a^2}\right) = \log_a 20 - \log_a(a^2).$$

Since

$$20 = 2^2 \times 5,$$

we obtain

$$\log_a 20 = 2\log_a 2 + \log_a 5.$$

Therefore,

$$= 2(0.301) + 0.699 - 2 = 0.602 + 0.699 - 2 = 1.301 - 2 = -0.699.$$

Thus,

$$\boxed{-0.699}.$$

4. Solution

Take logarithms of both sides:

$$\log(2^{x+1}) = \log(5^{x-1}).$$

Apply the power rule:

$$(x+1)\log 2 = (x-1)\log 5.$$

Expand:

$$x\log 2 + \log 2 = x\log 5 - \log 5.$$

Rearranging,

$$x(\log 2 - \log 5) = -(\log 5 + \log 2).$$

Hence,

$$x = \frac{\log 10}{\log 5 - \log 2} = \frac{1}{\log(5/2)}.$$

Therefore,

$$\boxed{x = \frac{1}{\log(5/2)} \approx 2.513.}$$

5. Solution

Use the product rule:

$$\log_3[x(x-6)] = 2.$$

Convert to exponential form:

$$x(x-6) = 3^2 = 9.$$

Thus,

$$x^2 - 6x - 9 = 0.$$

Apply the quadratic formula:

$$x = \frac{6 \pm \sqrt{36+36}}{2} = \frac{6 \pm \sqrt{72}}{2} = 3 \pm 3\sqrt{2}.$$

Check the domain:

$$x > 6.$$

Since

$$3 - 3\sqrt{2} < 0,$$

it is rejected.

Therefore, the solution is

$$\boxed{x = 3 + 3\sqrt{2}.}$$

3.3 Linear Equations and Inequalities

Linear Equations

Linear Equations in One Variable
A linear equation in one variable is an equation of degree one and can be written in the form

$$ax + b = 0,$$

where a and b are constants and $a \neq 0$.
The solution is obtained by isolating the variable:

$$x = -\frac{b}{a}.$$

Examples:

$$2x + 5 = 0, \qquad 3x - 6 = 0, \qquad -2x + 6 = 3x + 10.$$

Linear Equations in Two Variables
A linear equation in two variables has the form

$$ax + by = c,$$

where a, b, c are constants and $a, b \neq 0$.
Examples:

$$3x + 2y = 6, \qquad 5x + 6y + 2 = 0, \qquad -x + y = 5.$$

The graph of every linear equation in two variables is a straight line.

Slope of a Straight Line

The slope measures the steepness of a line.

If a line passes through (x_1, y_1) and (x_2, y_2), then

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Slope-Intercept Form

$$y = mx + b,$$

where

- m is the slope,

- b is the y-intercept.

Parallel Lines

Two lines are parallel if

$$m_1 = m_2.$$

Perpendicular Lines

Two lines are perpendicular if

$$m_1 m_2 = -1.$$

Linear Inequalities

A linear inequality in one variable has the form

$$ax + b < c,$$

or

$$ax + b \leq c,$$

or

$$ax + b > c,$$

or

$$ax + b \geq c.$$

The solution is usually expressed using interval notation.
Important Rule
Whenever both sides of an inequality are multiplied or divided by a negative number, the inequality sign reverses.
For example,
$-2x > 8$
implies
$x < -4$.

NGAT Quick Notes

- Linear equations have degree one.
- The graph of a linear equation is always a straight line.
- Parallel lines have equal slopes.
- Perpendicular lines have slopes whose product is -1.
- Reverse the inequality sign when multiplying or dividing by a negative number.
- Closed interval:

$$[a, b] = x : a \leq x \leq b$$

- Open interval:

$$(a, b) = x : a < x < b$$

- Half-open intervals:

$$[a, b), \qquad (a, b].$$

Solved Examples

1. Solve:

$$3x - 7 = 5x + 9$$

Solution

$$3x - 5x = 9 + 7$$

$$-2x = 16$$

$$x = -8$$

$$\boxed{x = -8}$$

2. Find the slope of the line passing through $(2, 3)$ and $(6, 11)$.

Solution

$$m = \frac{11 - 3}{6 - 2}$$

$$m = \frac{8}{4} = 2$$

$$\boxed{m = 2}$$

3. Solve the inequality

$$-3x + 6 > 12.$$

Solution

$$-3x > 6$$

Dividing by -3 reverses the inequality sign:

$$x < -2$$

$$\boxed{x < -2}$$

Self-Test Exercises

1. Solve: $4x + 5 = 21$

A. 2
B. 3
C. 4
D. 5

2. Solve: $7x - 3 = 4x + 12$

A. 3
B. 4
C. 5
D. 6

3. The slope of the line passing through $(1, 2)$ and $(5, 10)$ is

A. 1
B. 2
C. 3
D. 4

4. Which of the following lines is parallel to $y = 3x + 2$?

A. $y = 2x - 1$
B. $y = 3x - 5$
C. $y = -3x + 4$
D. $y = \frac{1}{3}x + 7$

5. A line perpendicular to the line $y = 2x + 5$ has slope

A. 2
B. $-\frac{1}{2}$
C. $\frac{1}{2}$
D. -2

6. Solve: $5x - 15 \leq 10$

A. $x \leq 5$
B. $x \geq 5$
C. $x \leq 1$
D. $x \geq 1$

7. Solve: $-2x + 4 > 10$

A. $x > 3$
B. $x < -3$
C. $x > 7$
D. $x < 7$

8. The y-intercept of $y = 4x - 7$ is

A. 4
B. -7
C. 7
D. -4

9. Which interval represents $2 < x \leq 7$?

A. $[2, 7]$
B. $(2, 7)$
C. $(2, 7]$
D. $[2, 7)$

10. Solve: $3(2x - 1) = 15$

A. 2
B. 3
C. 4
D. 5

Answer Key

1. C 2. C 3. B 4. B 5. B 6. A 7. B 8. B 9. C 10. B

3.4 Sequence and series

Sequences

Basic Terms

- A sequence is an ordered list of numbers.
- Each element is called a term.
- The general term is denoted by a_n.
- A sequence may be finite or infinite.

Arithmetic Sequence

- An arithmetic sequence is a sequence in which each terms except the first is obtained by adding a fixed number(positive/negative) to the preceding term. The fixed number is called common difference of the sequence.
- If A_n is an arithmetic sequence with first term A_1 and common difference d, then the n^{th} term of the sequence is given by:

$$A_n = A_1 + (n-1)d$$

- The common difference d is given by

$$d = A_{n+1} - A_n$$

Example:- $2, 5, 8, 11, \ldots$

Geometric Sequence

- A geometric sequence is a sequence in which each term is obtained by multiplying the previous term by a constant called the common ratio.
- If G_n is a geometric sequence then, the n^{th} term of the geometric sequence is given by:

$$G_n = G_1 r^{n-1}$$

- The common ratio r is also given by:

$$r = \frac{G_{n+1}}{G_n}, \quad G_n \neq 0$$

Example

$$2, 6, 18, 54, \ldots$$

Other Sequences (Pattern Identification)

- Apart from arithmetic and geometric progressions, many GAT questions feature non-standard sequences. For these, candidates are expected to analyze the relationship between consecutive terms to identify the structural rule of the sequence.

- Common sequence patterns to look for include:

  - **Perfect Squares/Cubes:** Terms are generated by squaring or cubing consecutive integers (n^2 or n^3), or by applying a fixed shift (e.g., n^2+1).
  - **Recursive (Fibonacci-type):** Each term is obtained by performing an operation on the immediate preceding terms (e.g., $a_n = a_{n-1} + a_{n-2}$).
  - **Alternating Sequences:** Two separate patterns interleaved into a single sequence (e.g., odd positions follow one rule, even positions follow another).
  - **Changing Differences:** The difference between terms is not constant, but the differences themselves form an arithmetic or geometric sequence.

Example 1 (Square Pattern with Shift)

$$2, 5, 10, 17, 26, \ldots$$

Rule: Each term corresponds to $n^2 + 1$ *(where* $n = 1, 2, 3, \ldots$*).*

Example 2 (Fibonacci Sequence)

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

Rule: Each term after the second is the sum of the two preceding terms.

Example 3 (Alternating Rule)

$$3, 10, 5, 8, 7, 6, \ldots$$

Rule: Odd terms increase by 2 $(3, 5, 7, \ldots)$*, while even terms decrease by 2* $(10, 8, 6, \ldots)$*.*

Sum of Arithmetic and Geometric Sequences

☞ Let A_1 be the first term and A_n the n^{th} term.

The sum of first n terms is given by:

$$S_n = \frac{n}{2}(A_1 + A_n)$$

Alternatively, if d is the common difference:

$$S_n = \frac{n}{2}\left[2A_1 + (n-1)d\right]$$

☞ Let G_1 be the first term and G_n the n^{th} term of a geometric sequence.

Sum of first n terms:

If $r \neq 1$ (common ratio):

$$S_n = G_1 \frac{1 - r^n}{1 - r}$$

Using the n^{th} term

$$S_n = \frac{G_1 - G_n r}{1 - r}$$

Special case: If $r = 1$,

$$S_n = nG_1$$

3.5 Solved Problems

1. The 23rd term of the sequence $-4, -2.5, -1, 0.5, \ldots$ is:

   A. 27.5
   B. 31
   C. 29
   D. 24

   Solution:
   The given sequence is: $-4, -2.5, -1, 0.5, \ldots$
   The common difference is:

   $$d = -2.5 - (-4) = 1.5.$$

   Thus, the sequence is an arithmetic progression (A.P.) with $a = -4$ and $d = 1.5$.
   The nth term of an arithmetic sequence is given by:

   $$a_n = a + (n-1)d.$$

Substituting $n = 23$:

$$a_{23} = -4 + (23 - 1)(1.5)$$
$$a_{23} = -4 + 22(1.5)$$
$$a_{23} = -4 + 33$$
$$a_{23} = 29.$$

$$\boxed{a_{23} = 29}$$

Answer: C) 29

2. Find the sum of the first 20 terms of the arithmetic sequence: $5, 9, 13, 17, \ldots$

   A. 860

   B. 900

   C. 780

   D. 830

   Solution:
   The first term is $A_1 = 5$ and the common difference is $d = 9 - 5 = 4$.
   We need to calculate the sum of the first $n = 20$ terms using the formula:

$$S_n = \frac{n}{2}[2A_1 + (n-1)d]$$

   Substituting our values:

$$S_{20} = \frac{20}{2}[2(5) + (20-1)4]$$
$$S_{20} = 10[10 + (19 \times 4)]$$
$$S_{20} = 10[10 + 76]$$
$$S_{20} = 10 \times 86 = 860.$$

$$\boxed{S_{20} = 860}$$

   Answer: A) 860

3. Find the 6th term of the geometric sequence: $3, 6, 12, 24, \ldots$

   A. 48

   B. 96

   C. 192

   D. 144

   Solution:
   The first term is $G_1 = 3$.
   The common ratio is $r = \frac{6}{3} = 2$.
   The formula for the nth term of a geometric progression is:

$$G_n = G_1 r^{n-1}$$

Substituting $n = 6$:

$$G_6 = 3 \times 2^{6-1}$$
$$G_6 = 3 \times 2^5$$
$$G_6 = 3 \times 32 = 96.$$

$$\boxed{G_6 = 96}$$

Answer: B) 96

4. What is the next term in the sequence: $0, 7, 26, 63, 124, \ldots$?

   A. 175

   B. 215

   C. 243

   D. 255

   Solution:
   Let us analyze the terms of the sequence: $0, 7, 26, 63, 124, \ldots$
   Notice that each term is remarkably close to a perfect cube $(1, 8, 27, 64, 125, \ldots)$. We can express the general term a_n by applying a fixed downward shift of 1 to consecutive cubes:

$$a_1 = 1^3 - 1 = 1 - 1 = 0$$
$$a_2 = 2^3 - 1 = 8 - 1 = 7$$
$$a_3 = 3^3 - 1 = 27 - 1 = 26$$
$$a_4 = 4^3 - 1 = 64 - 1 = 63$$
$$a_5 = 5^3 - 1 = 125 - 1 = 124$$

   Following this structural rule, the 6th term $(n = 6)$ must be:

$$a_6 = 6^3 - 1 = 216 - 1 = 215$$

$$\boxed{a_6 = 215}$$

   Answer: B) 215

5. Find the missing value (x) in the sequence: $2, 3, 5, 8, 13, x, 34, \ldots$

   A. 19

   B. 20

   C. 21

   D. 22

   Solution:
   Let us check the progression of terms: $2, 3, 5, 8, 13, x, 34$.
   If we check the differences, they are $1, 2, 3, 5$. This indicates a recursive dependency.

Let us test a Fibonacci-type rule where each term after the second is the sum of its two immediate preceding terms ($a_n = a_{n-1} + a_{n-2}$):

$$a_3 = a_2 + a_1 = 3 + 2 = 5$$
$$a_4 = a_3 + a_2 = 5 + 3 = 8$$
$$a_5 = a_4 + a_3 = 8 + 5 = 13$$

Therefore, the missing term x (a_6) must satisfy:

$$x = a_5 + a_4 = 13 + 8 = 21$$

To verify, check the subsequent term (a_7):

$$a_7 = a_6 + a_5 = 21 + 13 = 34$$

Since the rule consistently holds true, we find:

$$\boxed{x = 21}$$

Answer: C) 21

6. What is the next number in the alternating sequence: $2, 20, 4, 17, 8, 14, 16, \ldots$?

   A. 11
   B. 12
   C. 20
   D. 32

   Solution:
   The sequence fluctuates heavily upward and downward: $2, 20, 4, 17, 8, 14, 16, \ldots$. This behavior strongly implies an *Alternating Sequence* consisting of two interleaved patterns. Let us split the positions:

   - **Odd positions ($1^{\text{st}}, 3^{\text{rd}}, 5^{\text{th}}, 7^{\text{th}}$):** $2, 4, 8, 16, \ldots$
     Rule: Each term is multiplied by 2 (Geometric: $\times 2$).
   - **Even positions ($2^{\text{nd}}, 4^{\text{th}}, 6^{\text{th}}, \ldots$):** $20, 17, 14, \ldots$
     Rule: Each term decreases by 3 (Arithmetic: -3).

   The term we are looking for falls on the 8^{th} position, which is an even position. Following the even position rule:

   $$a_8 = a_6 - 3 = 14 - 3 = 11$$

   $$\boxed{a_8 = 11}$$

   Answer: A) 11

7. Find the next term in the sequence: $2, 3, 6, 11, 18, 27, \ldots$

   A. 34
   B. 36

C. 38

D. 40

Solution:
Let us examine the differences between consecutive terms in the sequence: $2, 3, 6, 11, 18, 27, \ldots$

$$\begin{aligned} 3 - 2 &= 1 \\ 6 - 3 &= 3 \\ 11 - 6 &= 5 \\ 18 - 11 &= 7 \\ 27 - 18 &= 9 \end{aligned}$$

The differences between consecutive terms are $1, 3, 5, 7, 9$. The difference is not constant, but the differences themselves form an arithmetic sequence of consecutive odd numbers increasing by $+2$ each step.
Therefore, the next difference must be $9 + 2 = 11$. Adding this difference to the last term yields:

$$a_7 = 27 + 11 = 38$$

$$\boxed{a_7 = 38}$$

Answer: C) 38

8. What is the next number in the sequence: $2, 3, 7, 16, 32, \ldots$?

A. 48

B. 52

C. 57

D. 64

Solution:
Let us analyze the differences between consecutive terms for $2, 3, 7, 16, 32, \ldots$:

$$\begin{aligned} 3 - 2 &= 1 \\ 7 - 3 &= 4 \\ 16 - 7 &= 9 \\ 32 - 16 &= 16 \end{aligned}$$

The first-level differences are $1, 4, 9, 16$. Notice that these differences are not random; they form a geometric sub-pattern of consecutive perfect squares $(1^2, 2^2, 3^2, 4^2)$. Following this structural rule, the next difference must be $5^2 = 25$. Adding 25 to the last term gives:

$$a_6 = 32 + 25 = 57$$

$$\boxed{a_6 = 57}$$

Answer: C) 57

Partial Sums and Series

☞ **Definition of an Infinite Series:** Given an infinite sequence of numbers $A_1, A_2, A_3, \dots, A_n, \dots$, an **infinite series** is the formal expression representing the sum of all terms in the sequence. It is denoted using sigma notation as:

$$\sum_{n=1}^{\infty} A_n = A_1 + A_2 + A_3 + \cdots + A_n + \dots$$

☞ **Definition of Partial Sums:** Because an infinite number of terms cannot be added directly, we analyze the series using its finite components. The n^{th} **partial sum**, denoted S_n, is the sum of the first n terms of the sequence:

$$S_n = \sum_{k=1}^{n} A_k = A_1 + A_2 + A_3 + \cdots + A_n$$

The set of all partial sums forms a new sequence $\{S_1, S_2, S_3, \dots, S_n, \dots\}$, known as the **sequence of partial sums**.

☞ **Convergence of a Series:** The sum (or value) of an infinite series is defined as the limit of its sequence of partial sums as n approaches infinity:

$$\sum_{n=1}^{\infty} A_n = \lim_{n \to \infty} S_n$$

If this limit exists and equals a finite number S, the series is said to **converge** to S. If the limit does not exist or equals $\pm\infty$, the series **diverges**.

3.6 Examples

1. An auditorium has 20 seats in the front row, 23 seats in the second row, 26 seats in the third row, and so on. If the auditorium has a total of 30 rows, what is the total seating capacity?

   A. $1,900$

   B. $1,915$

   C. $1,920$

   D. $1,935$

   Solution:
   Let us analyze the seating capacities across the rows as a finite sequence. This pattern forms an arithmetic progression where the first term is $A_1 = 20$, the common difference is $d = 3$, and the total number of terms (rows) is $n = 30$. Using the

arithmetic partial sum formula:

$$\begin{aligned}
S_n &= \frac{n}{2}\left[2A_1 + (n-1)d\right] \\
S_{30} &= \frac{30}{2}\left[2(20) + (30-1)3\right] \\
S_{30} &= 15\left[40 + 87\right] \\
S_{30} &= 15 \times 127 = 1935
\end{aligned}$$

$$\boxed{S_{30} = 1935}$$

Answer: D) 1,935

2. A certain type of bacteria splits into two every hour. If a culture starts with 5 bacteria, which of the following expressions represents the total cumulative number of bacteria generated up to the end of the 10^{th} hour (including the initial batch)?

   A. $5(2^{10} - 1)$
   B. $5(2^{11} - 1)$
   C. $10(2^{10} - 1)$
   D. $5(2^{10})$

   Solution:
   Let us track the total count at every hour mark. The terms form a geometric sequence: $5, 10, 20, \ldots$ where each term represents a milestone count. Counting the initial state (hour 0) through the end of hour 10 yields a total of 11 distinct terms. Here, the first term is $G_1 = 5$ and the common ratio is $r = 2$. Using the geometric partial sum formula for $r \neq 1$:

$$\begin{aligned}
S_n &= G_1 \frac{1 - r^n}{1 - r} \\
S_{11} &= 5 \cdot \frac{1 - 2^{11}}{1 - 2} \\
S_{11} &= 5 \cdot \frac{1 - 2^{11}}{-1} \\
S_{11} &= 5(2^{11} - 1)
\end{aligned}$$

$$\boxed{S_{11} = 5(2^{11} - 1)}$$

   Answer: B) $5(2^{11} - 1)$

3. The sequence of partial sums of an infinite series $\sum_{n=1}^{\infty} a_n$ is given by the formula $S_n = \frac{3n^2+5}{2n^2-n}$. What value does the infinite series converge to?

   A. 0
   B. $\frac{3}{2}$
   C. 3
   D. The series diverges

Solution:
By analytical definition, the total sum of an infinite series is equivalent to the mathematical limit of its matching sequence of partial sums as the variable n approaches infinity. Let us evaluate this limit directly:

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \lim_{n\to\infty} S_n \\ &= \lim_{n\to\infty} \frac{3n^2+5}{2n^2-n} \\ &= \lim_{n\to\infty} \frac{3+\frac{5}{n^2}}{2-\frac{1}{n}} \\ &= \frac{3+0}{2-0} = \frac{3}{2}\end{aligned}$$

Since the limit resolves cleanly to a specific, finite value, the series is convergent.

$$\boxed{\lim_{n\to\infty} S_n = \frac{3}{2}}$$

Answer: B) $\frac{3}{2}$

4. If the n^{th} partial sum of a series is known to be $S_n = 1 - \frac{1}{n!}$, what is the value of the third term, a_3, of the original sequence?

   A. $\frac{5}{6}$
   B. $\frac{1}{6}$
   C. $\frac{1}{3}$
   D. $\frac{1}{24}$

   Solution:
   Let us make use of the fundamental identity connecting individual sequence terms to structural partial sums, which states that $a_n = S_n - S_{n-1}$. To target the value of the third term a_3, we find S_3 and S_2:

$$\begin{aligned}S_3 &= 1 - \frac{1}{3!} = 1 - \frac{1}{6} = \frac{5}{6} \\ S_2 &= 1 - \frac{1}{2!} = 1 - \frac{1}{2} = \frac{1}{2} \\ a_3 &= S_3 - S_2 \\ a_3 &= \frac{5}{6} - \frac{1}{2} = \frac{5-3}{6} = \frac{2}{6} = \frac{1}{6}\end{aligned}$$

$$\boxed{a_3 = \frac{1}{6}}$$

   Answer: B) $\frac{1}{6}$

5. The sum of the first n terms of a specific series is given by the formula $S_n = 2n^2+3n$. What is the general expression for the n^{th} term, a_n, of the underlying sequence?

A. $4n + 1$
B. $4n + 3$
C. $2n + 1$
D. $4n - 1$

Solution:
To recover the general term a_n from a known partial sum formula S_n, we apply the fundamental algebraic relationship $a_n = S_n - S_{n-1}$ for all $n > 1$:

$$\begin{aligned} S_n &= 2n^2 + 3n \\ S_{n-1} &= 2(n-1)^2 + 3(n-1) \\ &= 2(n^2 - 2n + 1) + 3n - 3 \\ &= 2n^2 - 4n + 2 + 3n - 3 \\ &= 2n^2 - n - 1 \end{aligned}$$

Now, we subtract S_{n-1} from S_n to isolate a_n:

$$\begin{aligned} a_n &= (2n^2 + 3n) - (2n^2 - n - 1) \\ &= 2n^2 + 3n - 2n^2 + n + 1 \\ &= 4n + 1 \end{aligned}$$

Let us quickly check for $n = 1$: $S_1 = 2(1)^2 + 3(1) = 5$, and $a_1 = 4(1) + 1 = 5$. The formula holds true.

$$\boxed{a_n = 4n + 1}$$

Answer: A) 4n + 1

6. An infinite geometric series has a first term $G_1 = 12$ and its sequence of partial sums converges to a total value of 16. What is the common ratio r of this series?

A. $\frac{1}{4}$
B. $\frac{1}{3}$
C. $\frac{1}{2}$
D. $\frac{3}{4}$

Solution:
By definition, the value to which a sequence of partial sums converges is the sum of the infinite series (S). For an infinite geometric series to converge, the absolute value of its common ratio must satisfy $|r| < 1$. The sum is given by the formula $S = \frac{G_1}{1-r}$. Substituting our known parameters:

$$\begin{aligned} 16 &= \frac{12}{1-r} \\ 16(1-r) &= 12 \\ 16 - 16r &= 12 \\ 16 - 12 &= 16r \\ 4 &= 16r \\ r &= \frac{4}{16} = \frac{1}{4} \end{aligned}$$

Since $|1/4| < 1$, this confirms the validity of our convergence assumption.

$$\boxed{r = \frac{1}{4}}$$

Answer: A) $\frac{1}{4}$

3.7 Self test exercise

1. What is the common difference of an arithmetic sequence if its 4th term is 14 and its 11th term is 35?

   A. 2

   B. 3

   C. 4

   D. 5

2. Find the sum of the first 5 terms of the geometric sequence: $2, -6, 18, -54, \ldots$

   A. 122

   B. -122

   C. 244

   D. -244

3. An infinite geometric series has a first term $G_1 = 12$ and a common ratio $r = \frac{1}{3}$. What value does the total sum of this sequence approach?

   A. 16

   B. 18

   C. 24

   D. 36

4. What is the 8th term of the geometric sequence $3, 6, 12, 24, \ldots$?

   A. 192

   B. 384

   C. 768

   D. 1536

5. Find the sum of the first 20 terms of the arithmetic series $5 + 9 + 13 + 17 + \ldots$

   A. 860

   B. 910

   C. 930

   D. 1430

6. An infinite geometric series has a first term $a_1 = 12$ and a common ratio $r = \frac{1}{3}$. What is its sum?

   A. 16

   B. 18

   C. 24

   D. 36

7. If the third term of a geometric sequence is 36 and the sixth term is 972, what is the first term?

   A. 2

   B. 3

   C. 4

   D. 6

8. How many terms are there in the finite arithmetic sequence $2, 7, 12, 17, \ldots, 112$?

   A. 21

   B. 22

   C. 23

   D. 24

9. Find the sum of the first 6 terms of the geometric series $4 + 12 + 36 + \ldots$

   A. 728

   B. 1456

   C. 2184

   D. 4368

10. Find the value of x such that $x-3$, $x+1$, and $3x-1$ form the first three consecutive terms of an arithmetic sequence.

    A. 2

    B. 3

    C. 4

    D. 6

Answers & Explanations to Self Test:

1. **B) 3** — Using the formula $A_n = A_1 + (n-1)d$, we can set up a system of linear equations:

$$A_4 = A_1 + 3d = 14$$

$$A_{11} = A_1 + 10d = 35$$

Subtracting the first equation from the second eliminates A_1:

$$(A_1 + 10d) - (A_1 + 3d) = 35 - 14$$

$$7d = 21 \implies d = 3.$$

2. **A) 122** — Here $G_1 = 2$ and the common ratio $r = \frac{-6}{2} = -3$. Since $r \neq 1$, we use the geometric sum formula for $n = 5$:

$$S_5 = G_1 \frac{1 - r^5}{1 - r} = 2 \times \frac{1 - (-3)^5}{1 - (-3)}$$

$$S_5 = 2 \times \frac{1 - (-243)}{4} = 2 \times \frac{244}{4} = \frac{488}{4} = 122.$$

3. **B) 18** — For an infinite geometric sequence where the ratio is a fraction $|r| < 1$, the infinite sum approaches the limit formula $S_\infty = \frac{G_1}{1-r}$. Substituting our values yields:

$$S_\infty = \frac{12}{1 - \frac{1}{3}} = \frac{12}{\frac{2}{3}} = 12 \times \frac{3}{2} = 18.$$

4. **Correct Answer: B**
The first term is $a_1 = 3$ and the common ratio is $r = \frac{6}{3} = 2$. Using the geometric sequence formula $a_n = a_1 \cdot r^{n-1}$:

$$a_8 = 3 \cdot 2^{8-1}$$

$$a_8 = 3 \cdot 2^7 = 3 \cdot 128 = 384$$

5. **Correct Answer: A**
For this arithmetic series, $a_1 = 5$, $d = 4$, and $n = 20$. Using the sum formula $S_n = \frac{n}{2}[2a_1 + (n-1)d]$:

$$S_{20} = \frac{20}{2}[2(5) + (20 - 1)4]$$

$$S_{20} = 10[10 + (19)4] = 10[10 + 76]$$

$$S_{20} = 10 \cdot 86 = 860$$

6. **Correct Answer: B**
The sum of an infinite geometric series when $|r| < 1$ is given by $S = \frac{a_1}{1-r}$. Substituting $a_1 = 12$ and $r = \frac{1}{3}$:

$$S = \frac{12}{1 - \frac{1}{3}} = \frac{12}{\frac{2}{3}}$$

$$S = 12 \cdot \frac{3}{2} = 18$$

7. **Correct Answer: C**
Using the formula $a_n = a_1 \cdot r^{n-1}$, we can set up the ratio of the two terms:

$$\frac{a_6}{a_3} = \frac{a_1 r^5}{a_1 r^2} \implies \frac{972}{36} = r^3$$

$$27 = r^3 \implies r = 3$$

Substitute $r = 3$ back into the expression for a_3:

$$a_3 = a_1 \cdot r^2 \implies 36 = a_1 \cdot 3^2$$

$$36 = 9a_1 \implies a_1 = 4$$

8. **Correct Answer: C**
Here, the first term is $a_1 = 2$, the common difference is $d = 7 - 2 = 5$, and the last term is $a_n = 112$. Using the formula $a_n = a_1 + (n-1)d$:

$$112 = 2 + (n-1)5$$

$$110 = 5(n-1)$$

$$22 = n - 1 \implies n = 23$$

9. **Correct Answer: B**
The first term is $a_1 = 4$ and the common ratio is $r = \frac{12}{4} = 3$. Using the finite geometric series sum formula $S_n = \frac{a_1(r^n-1)}{r-1}$:

$$S_6 = \frac{4(3^6 - 1)}{3 - 1}$$

$$S_6 = \frac{4(729 - 1)}{2} = 2 \cdot 728 = 1456$$

10. **Correct Answer: C**
For three terms a, b, and c to form an arithmetic sequence, the difference between consecutive terms must be equal. Therefore, we can set up the relation:

$$b - a = c - b \implies 2b = a + c$$

Substituting our given terms $a = x - 3$, $b = x + 1$, and $c = 3x - 1$:

$$2(x + 1) = (x - 3) + (3x - 1)$$

Distribute and combine like terms on both sides:

$$2x + 2 = 4x - 4$$

Isolate the variable x by rearranging the terms:

$$2 + 4 = 4x - 2x$$

$$6 = 2x \implies x = 3$$

3.8 Relations and Functions

Relation and Function

- ☞ Let A and B be two sets. A *relation* R from A to B is a subset of the Cartesian product $A \times B$, i.e.,

$$R \subseteq A \times B.$$

If $(a, b) \in R$, we write aRb.

- ☞ **Domain and Range:**
  - Domain: $\{a \in A : (a, b) \in R$ for some $b \in B\}$
  - Range: $\{b \in B : (a, b) \in R$ for some $a \in A\}$

- ☞ **Types of Relations**

  Let R be a relation on a set A.
  - **Reflexive:** $\forall a \in A,\ (a, a) \in R$
  - **Symmetric:** If $(a, b) \in R$, then $(b, a) \in R$
  - **Antisymmetric:** If $(a, b) \in R$ and $(b, a) \in R$, then $a = b$
  - **Transitive:** If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$

- ☞ A function f from a set A to a set B is a relation that assigns to each element of A exactly one element of B.

$$f : A \to B$$

- ☞ **Domain, Codomain, Range:**
  - Domain: Set A
  - Codomain: Set B
  - Range: $\{f(a) : a \in A\}$

- ☞ **Types of Functions**
  - **One-to-one (Injective):** $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$
  - **Onto (Surjective):** For every $b \in B$, there exists $a \in A$ such that $f(a) = b$
  - **Bijective:** Function is both injective and surjective
  - **Constant Function:** $f(a) = c$ for all $a \in A$
  - **Identity Function:** $f(a) = a$ for all $a \in A$

3.9 Examples

1. Let $A = \{1, 2, 3\}$. A relation R defined on set A is given by $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$. Which of the following best describes the properties of R?

   A. Reflexive and Symmetric, but not Transitive

   B. Reflexive and Transitive, but not Symmetric

   C. An Equivalence Relation

   D. Symmetric and Transitive, but not Reflexive

   Solution:
   Let us evaluate the three core properties of an equivalence relation step by step:

   - **Reflexivity:** For R to be reflexive, every element $x \in A$ must satisfy $(x, x) \in R$. Since $(1,1), (2,2)$, and $(3,3)$ are all present in R, the relation is **reflexive**.
   - **Symmetry:** For R to be symmetric, if $(x, y) \in R$, then $(y, x) \in R$. The non-identity pairs are $(1, 2)$ and $(2, 1)$. Since both elements are present, the relation is **symmetric**.
   - **Transitivity:** For R to be transitive, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$. Let us check the links:

     $$(1,2) \in R \text{ and } (2,1) \in R \implies (1,1) \in R \quad \text{(True)}$$
     $$(2,1) \in R \text{ and } (1,2) \in R \implies (2,2) \in R \quad \text{(True)}$$

     All composition paths hold true, meaning the relation is **transitive**.

   Because R is simultaneously reflexive, symmetric, and transitive, it qualifies as an equivalence relation.

   $$\boxed{R \text{ is an Equivalence Relation}}$$

   Answer: C) An Equivalence Relation

2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$. Which statement accurately describes the mathematical classification of this function?

   A. Injective but not Surjective

   B. Surjective but not Injective

   C. Bijective (Both Injective and Surjective)

   D. Neither Injective nor Surjective

   Solution:
   Let us separate the analytical testing into two parts: injectivity (one-to-one) and surjectivity (onto).

Part 1: Injectivity
Assume that $f(x_1) = f(x_2)$ for any arbitrary inputs $x_1, x_2 \in \mathbb{R}$:

$$\begin{aligned} 3x_1 - 5 &= 3x_2 - 5 \\ 3x_1 &= 3x_2 \\ x_1 &= x_2 \end{aligned}$$

Since $f(x_1) = f(x_2)$ strictly implies $x_1 = x_2$, the function is completely **injective**.
Part 2: Surjectivity
Let y be any real number in the codomain $\mathbb{R}$. We set up the equation $y = f(x)$ and solve for the input parameter x:

$$\begin{aligned} y &= 3x - 5 \\ y + 5 &= 3x \\ x &= \frac{y+5}{3} \end{aligned}$$

Since $\frac{y+5}{3}$ is a valid real number for any choice of $y \in \mathbb{R}$, every target value has a valid corresponding domain input. Thus, the function is **surjective**.

Because the mapping is both injective and surjective, it is classified as a bijection.

$$\boxed{f(x) \text{ is Bijective}}$$

Answer: C) Bijective (Both Injective and Surjective)

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined on the set of natural numbers by $f(x) = x^2$. Why does this specific function fail to be surjective?

   A. Because negative inputs are not allowed in the domain.
   B. Because certain elements in the codomain, like 2 or 3, have no matching natural number pre-images.
   C. Because different inputs map to identical outputs.
   D. The function does not fail; it is surjective.

   Solution:
   By definition, a function $f : X \to Y$ is surjective if its range matches its codomain exactly, meaning that for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. Let us evaluate this function over the natural numbers domain $\mathbb{N} = \{1, 2, 3, \dots\}$:

   $$\begin{aligned} \text{Range}(f) &= \{1^2, 2^2, 3^2, 4^2, \dots\} \\ &= \{1, 4, 9, 16, \dots\} \end{aligned}$$

   Let us choose an element from the codomain $\mathbb{N}$, for example, $y = 2$. We check for its pre-image:

   $$\begin{aligned} x^2 &= 2 \\ x &= \sqrt{2} \end{aligned}$$

Since $\sqrt{2} \notin \mathbb{N}$, the element 2 has no valid input pre-image in this system. Because there are leftover values in the target set that are never mapped to, the function fails surjectivity.

$$\boxed{2 \in \text{Codomain has no pre-image } x \in \mathbb{N}}$$

Answer: B) Because certain elements in the codomain, like 2 or 3, have no matching natural number pre-images.

Self test exercise

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. What is the total number of distinct relations that can be defined from set A to set B?

   A. 6

   B. 8

   C. 32

   D. 64

2. Let R be a relation on the set of integers $\mathbb{Z}$ defined by xRy if and only if $x - y$ is divisible by 5. Which of the following properties does R satisfy?

   A. Reflexive and Symmetric only

   B. Symmetric and Transitive only

   C. Equivalence Relation

   D. Partial Order Relation

3. If a function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x - 5$, what type of function is f?

   A. Injective (One-to-One) but not Surjective (Onto)

   B. Surjective (Onto) but not Injective (One-to-One)

   C. Bijective (Both One-to-One and Onto)

   D. Neither Injective nor Surjective

4. Find the domain of the real-valued function $f(x) = \frac{\sqrt{4-x^2}}{x-1}$.

   A. $[-2, 2]$

   B. $[-2, 1) \cup (1, 2]$

   C. $(-\infty, 2]$

   D. $[-2, 1) \cup (1, \infty)$

5. Let $f(x) = 2x + 3$ and $g(x) = x^2 - 1$ be functions from $\mathbb{R}$ to $\mathbb{R}$. What is the value of the composite function $(g \circ f)(-2)$?

   A. 0

   B. 3

   C. -1

D. 2

6. Let $R = \{(1,2),(2,3),(3,1)\}$ be a relation on the set $A = \{1,2,3\}$. What is the minimum number of ordered pairs that must be added to R to make it a transitive relation?

   A. 2

   B. 3

   C. 4

   D. 6

7. If $f(x) = \frac{2x+1}{x-3}$ for $x \neq 3$, which of the following expressions represents the inverse function $f^{-1}(x)$?

   A. $\frac{3x+1}{x-2}$

   B. $\frac{3x-1}{x+2}$

   C. $\frac{2x-1}{x-3}$

   D. $\frac{x-3}{2x+1}$

8. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the property $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Which of the following functions is an example of such an odd function?

   A. $f(x) = x^2 + \cos(x)$

   B. $f(x) = x^3 + \sin(x)$

   C. $f(x) = e^x$

   D. $f(x) = |x| - 4$

9. Let A be a set containing n distinct elements. How many distinct bijective functions can be formed from A to itself?

   A. n^2

   B. 2^n

   C. $n!$

   D. n^n

10. What is the range of the real-valued function $f(x) = \frac{x^2}{1+x^2}$?

    A. $[0, \infty)$

    B. $[0, 1]$

    C. $[0, 1)$

    D. $(-\infty, 1)$

Answer Key & Explanations

1. **Correct Answer: D**
The number of elements in the Cartesian product $A \times B$ is given by $|A \times B| = |A| \times |B| = 3 \times 2 = 6$. A relation from A to B is any subset of $A \times B$. Therefore, the total number of distinct relations is the number of subsets of $A \times B$, which is $2^6 = 64$.

2. **Correct Answer: C**
The relation is an equivalence relation because:
   - **Reflexive**: $x - x = 0$, which is divisible by 5, so xRx holds.
   - **Symmetric**: If xRy, then $x - y = 5k$ for some integer k. Thus, $y - x = -5k = 5(-k)$, meaning yRx holds.
   - **Transitive**: If xRy and yRz, then $x - y = 5k$ and $y - z = 5m$. Adding these equations yields $x - z = 5(k + m)$, meaning xRz holds.

3. **Correct Answer: C**
To check injectivity: $3x_1 - 5 = 3x_2 - 5 \implies x_1^3 = x_2^3 \implies x_1 = x_2$, which means it is injective. To check surjectivity: For any $y \in \mathbb{R}$, we can set $y = 3x - 5 \implies x = \frac{y+5}{3}$, which always exists as a real number. Thus, f is both injective and surjective, making it a bijection.

4. **Correct Answer: B**
For the numerator to be real, the term inside the square root must be non-negative: $4 - x^2 \geq 0 \implies x^2 \leq 4 \implies -2 \leq x \leq 2$. For the denominator to be non-zero, $x - 1 \neq 0 \implies x \neq 1$. Combining these intervals gives $[-2, 1) \cup (1, 2]$.

5. **Correct Answer: A**
First, evaluate the inner function $f(-2)$:
$$f(-2) = 2(-2) + 3 = -4 + 3 = -1$$
Next, substitute this result into the outer function $g(x)$:
$$(g \circ f)(-2) = g(f(-2)) = g(-1) = (-1)^2 - 1 = 1 - 1 = 0$$

6. **Correct Answer: D**
For transitivity, if $(1, 2)$ and $(2, 3)$ are in R, then $(1, 3)$ must be added. If $(2, 3)$ and $(3, 1)$ are in R, then $(2, 1)$ must be added. If $(3, 1)$ and $(1, 2)$ are in R, then $(3, 2)$ must be added. Furthermore, adding these requires reflexivity loops like $(1, 1), (2, 2), (3, 3)$ to complete full transitivity cycles across all interconnected element chains. Thus, 6 ordered pairs are required to fulfill all elements of a transitive relation environment on A.

7. **Correct Answer: A**
Set $y = f(x)$ and solve for x:
$$y = \frac{2x + 1}{x - 3} \implies y(x - 3) = 2x + 1 \implies xy - 3y = 2x + 1$$

Rearrange to isolate x:

$$xy - 2x = 3y + 1 \implies x(y-2) = 3y + 1 \implies x = \frac{3y+1}{y-2}$$

Swapping variables gives $f^{-1}(x) = \frac{3x+1}{x-2}$.

8. **Correct Answer: B**
   An odd function satisfies $f(-x) = -f(x)$. Testing option B:

   $$f(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin(x) = -(x^3 + \sin(x)) = -f(x)$$

   Options A and D represent even functions ($f(-x) = f(x)$), and option C is neither.

9. **Correct Answer: C**
   A bijective function from a finite set A to itself is equivalent to a permutation of the n elements. The number of ways to arrange n distinct elements into n positions uniquely is $n \cdot (n-1) \cdot \cdots \cdot 1 = n!$.

10. **Correct Answer: C**
    Since $x^2 \geq 0$ for all real numbers, the numerator and denominator are always non-negative, meaning $f(x) \geq 0$. Also, because $x^2 < 1 + x^2$ for all x, the fraction $\frac{x^2}{1+x^2}$ is strictly less than 1. As $x \to \infty$, $f(x) \to 1$ but never reaches it. Thus, the range is $[0, 1)$.

4 Geometry

4.1 Polygons

A **polygon** is a closed, two-dimensional geometric figure formed by three or more straight line segments (sides) that intersect only at their endpoints (vertices).

- **Classification:** Polygons are named based on their number of sides (e.g., Triangle: 3 sides, Quadrilateral: 4 sides, Pentagon: 5 sides, Hexagon: 6 sides).
- **Regular Polygons:** A polygon is regular if all its sides are equal in length (equilateral) and all its interior angles are equal in measure (equiangular).
- **Interior Angles:** The sum of the interior angles of an n-sided polygon is given by the formula $(n-2) \times 180°$.
- **Exterior Angles:** The sum of the exterior angles of *any* convex polygon is always $360°$.

4.2 Perimeter, Area

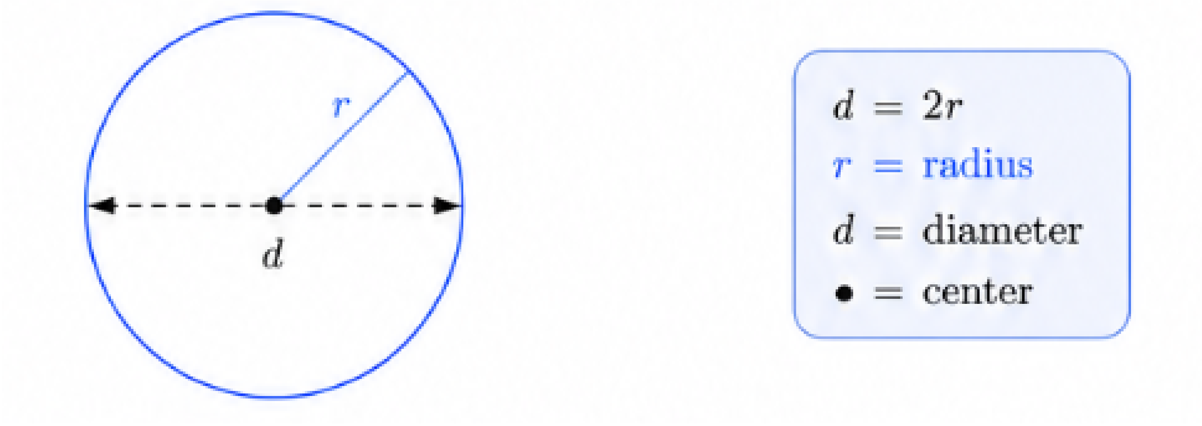
- **Perimeter:** The total continuous distance around the outside of a two-dimensional shape. For any polygon, it is the sum of the lengths of all its sides.
- **Circumference:** The specific term used for the perimeter of a circle (the distance around it).
- **Area:** The amount of two-dimensional space enclosed inside a shape, measured in square units.

4.3 Surface area, Volume

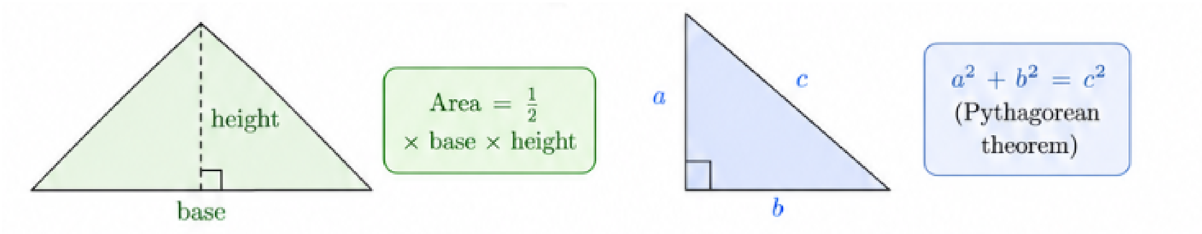
- **Solid Figures (3D Shapes):** Three-dimensional objects that have length, width, and depth (e.g., rectangular prisms, cylinders, spheres, cones).
- **Surface Area (SA):** The total area of all the outward-facing faces or surfaces of a three-dimensional object.
- **Volume (V):** The total amount of three-dimensional space an object occupies, measured in cubic units.

Short Notes

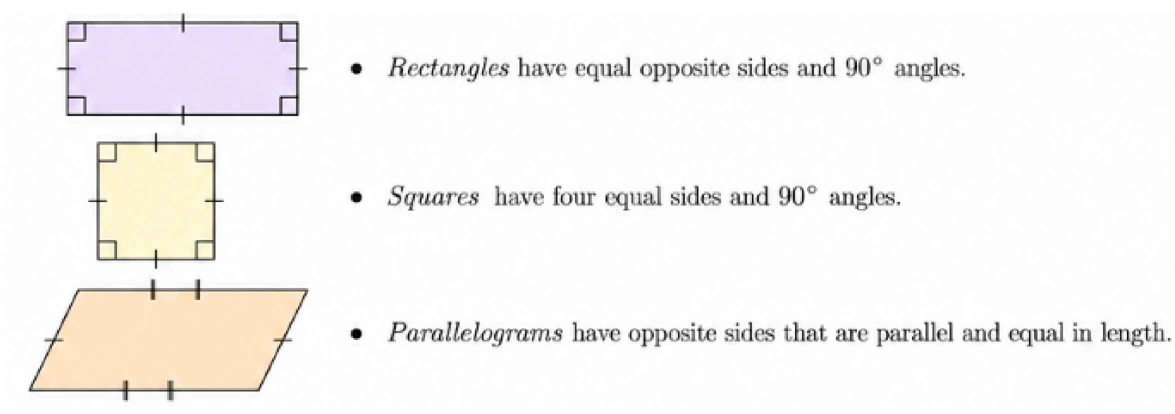
1. **Circles:** A circle is a set of all points in a plane that are at a fixed distance (radius, r) from a central point. The diameter (d) is twice the radius ($d = 2r$).



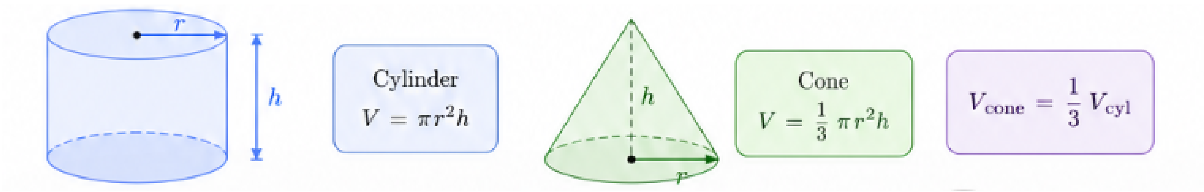
2. **Triangles:** The area of a triangle is always $\frac{1}{2} \times$ base $\times$ height. For a right-angled triangle, the Pythagorean theorem states $a^2 + b^2 = c^2$, where c is the hypotenuse.



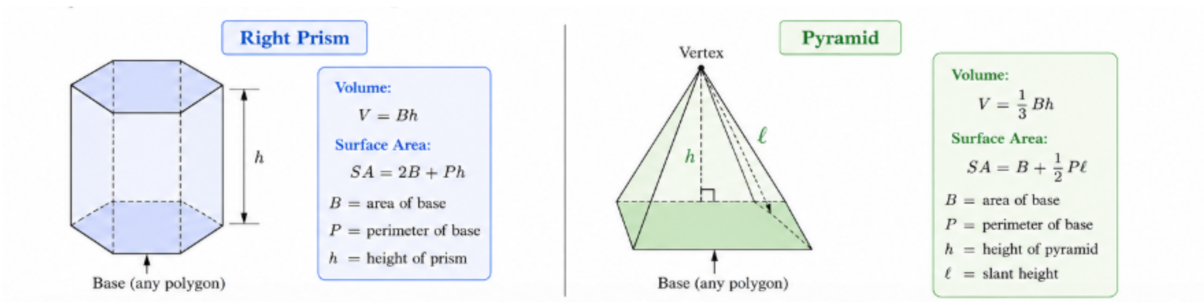
3. **Quadrilaterals:**
   - *Rectangles* have equal opposite sides and 90° angles.
   - *Squares* have four equal sides and 90° angles.
   - *Parallelograms* have opposite sides that are parallel and equal in length.



4. **Cylinders and Cones:** A cylinder is essentially a circular prism. A cone is exactly $\frac{1}{3}$ the volume of a cylinder with the same base radius and height.



5. **Prisms and Pyramids:** A prism has two congruent, parallel bases and rectangular lateral faces. A pyramid has one base and triangular lateral faces that meet at a single vertex.



Important Formulas

2D Shapes (Perimeter & Area)

- **Triangle:** Area $= \frac{1}{2}bh$
- **Rectangle:** Area $= l \times w$, Perimeter $= 2(l + w)$
- **Square:** Area $= s^2$, Perimeter $= 4s$
- **Parallelogram:** Area $= b \times h$
- **Circle:** Circumference $= 2\pi r = \pi d$, Area $= \pi r^2$

3D Shapes (Surface Area & Volume)

- **Rectangular Solid (Prism):** Volume $= l \times w \times h$, Surface Area $= 2(lw + lh + wh)$
- **Cube:** Volume $= s^3$, Surface Area $= 6s^2$
- **Cylinder:** Volume $= \pi r^2 h$, Surface Area $= 2\pi rh + 2\pi r^2$
- **Sphere:** Volume $= \frac{4}{3}\pi r^3$, Surface Area $= 4\pi r^2$

Rules to Remember

1. **The Sum of Polygon Angles:** Each interior angle of a *regular* polygon with n sides is:
$$\text{Interior Angle} = \frac{(n-2)\times 180^\circ}{n}$$
2. **The π Approximations:** In the GAT, you can generally approximate $\pi \approx 3.14$ or $\frac{22}{7}$, but look at the multiple-choice options first—many answers leave π in its symbolic form.
3. **Inscribed Shapes:** If a circle is inscribed inside a square, the diameter of the circle is equal to the side length of the square. If a square is inscribed inside a circle, the diagonal of the square is equal to the diameter of the circle.
4. **Unit Consistency:** Always check that dimensions are in the same units (e.g., converting cm to meters if required) before calculating areas or volumes.

4.4 Examples

1. What is the measure of each interior angle of a regular hexagon?

   Solution: A hexagon has $n = 6$ sides.

   Sum of interior angles $= (6-2)\times 180^\circ = 4\times 180^\circ = 720^\circ$.

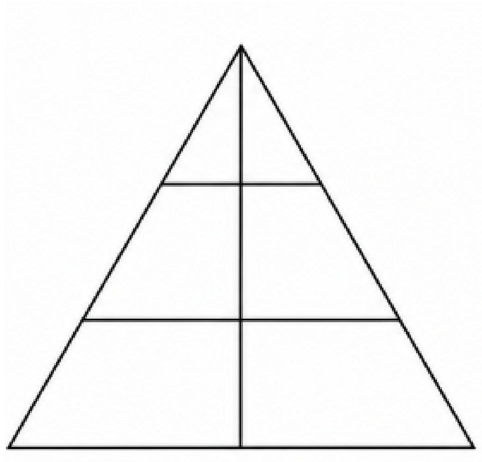
   Since it is a regular hexagon, divide the total sum by 6:
$$\text{Each angle} = \frac{720^\circ}{6} = 120^\circ$$
2. A circular track has a radius of 7 meters. A runner completes 5 full laps. How far did the runner travel? (Take $\pi = \frac{22}{7}$)

   Solution: Distance for one lap equals the circumference (C).
$$C = 2\pi r = 2\times\frac{22}{7}\times 7 = 44 \text{ meters}$$
   Total distance for 5 laps $= 44\times 5 = 220$ meters.

3. A cylinder has a base radius of 3 cm and a height of 10 cm. Find its total surface area in terms of π.

   Solution: Use the surface area formula: $\text{SA} = 2\pi rh + 2\pi r^2$.
$$\text{SA} = 2\pi(3)(10) + 2\pi(3)^2$$
$$\text{SA} = 60\pi + 2\pi(9) = 60\pi + 18\pi = 78\pi\ \text{cm}^2$$

4. How many triangles are there in the figure below?



A. 6

B. 9

C. 12

D. 18

Solution:
To solve this systematically, we can use a standard combinatorial formula for triangles that are split by vertical lines originating from the apex to the base.

First, count the number of base segments at the bottom level. The vertical line splits the base into 2 individual segments. The number of triangles formed on a single horizontal base is given by the sum of these segments:

$$1 + 2 = 3 \text{ triangles}$$

(These are the left small triangle, the right small triangle, and the combined large triangle).

Next, count the number of horizontal levels (parallel bases) in the figure. There are 3 horizontal lines acting as bases. Each horizontal line replicates the exact same vertical split pattern above it.

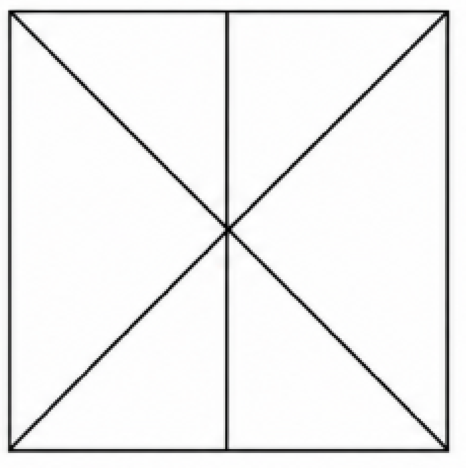
Therefore, the total number of triangles is:

$$\text{Total Triangles} = 3 \text{ (triangles per base)} \times 3 \text{ (horizontal bases)} = 9$$

$$\boxed{\text{Total} = 9}$$

Answer: B) 9

5. Find the total number of triangles in the following figure:



A. 4

B. 6

C. 8

D. 10

Solution:
Let us count the triangles by categorizing them by their size (the number of smaller component regions they contain):

- **Single-region triangles (Size 1):** The two intersecting diagonals divide the square into 4 small, non-overlapping individual triangles (Top, Bottom, Left, Right). $\rightarrow$ **4** triangles.
- **Two-region combined triangles (Size 2):** Combining two adjacent smaller triangles along a diagonal line forms a larger triangle.
  - Left + Top regions
  - Top + Right regions
  - Right + Bottom regions
  - Bottom + Left regions

  This accounts for another $\rightarrow$ **4** larger triangles.

Adding these together: $4 + 4 = 8$.

GAT Rule of Thumb: For any square or rectangle divided fully by its two diagonals, the total number of triangles is always equal to the number of individual small compartments multiplied by 2 ($4 \times 2 = 8$).

$$\boxed{\text{Total} = 8}$$

Answer: C) 8

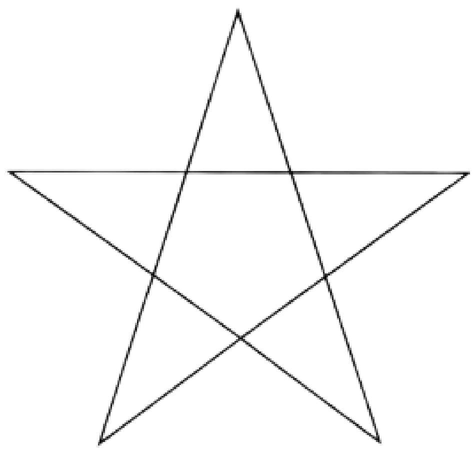
6. How many triangles can be found in the complex star polygon figure below?

   A. 5

   B. 8

   C. 10

   D. 12



Solution:
A standard 5-pointed star contains overlapping regions that require careful classification:

- **Outer Triangles (The points):** There are 5 prominent, non-overlapping small triangles forming the outer points of the star tip. $\rightarrow$ **5** triangles.
- **Inner Triangles (Large overlapping structures):** Look at the lines that skip a vertex to form the star body. Each of the 5 inner vertices of the central pentagon acts as a base point for a much larger triangle that stretches across the star's width. Since there are 5 such internal intersection points, there are exactly 5 of these large triangles. $\rightarrow$ **5** triangles.

Total = 5 (small tips) + 5 (large overlapping bodies) = 10 triangles.

$$\boxed{\text{Total} = 10}$$

Answer: C) 10

4.5 Self test exercise

1. Find the area of an equilateral triangle whose perimeter is 18 cm.

   (A) $6\sqrt{3}$ cm^2
   (B) $9\sqrt{3}$ cm^2
   (C) 18 cm^2
   (D) $36\sqrt{3}$ cm^2

2. If the area of a circle is 36π, what is its circumference?

   (A) 6π
   (B) 12π
   (C) 18π
   (D) 24π

3. A rectangular water tank is 5 meters long, 4 meters wide, and 3 meters deep. If it is filled to the top, what is the volume of the water in liters? (Note: 1 m^3 = 1000 liters)

   (A) 60 liters
   (B) 600 liters
   (C) $6,000$ liters
   (D) $60,000$ liters

4. What is the total sum of the exterior angles of an octagonal (8-sided) polygon?

   (A) $180°$
   (B) $360°$
   (C) $1080°$
   (D) $1440°$

Answers & Explanations to Self Test:

1. **B** — Since perimeter is 18 cm, each side of the equilateral triangle is $\frac{18}{3} = 6$ cm. The area formula for an equilateral triangle is $\frac{s^2\sqrt{3}}{4}$. Thus, $\frac{36\sqrt{3}}{4} = 9\sqrt{3}$ cm^2.

2. **B** — Area $= \pi r^2 = 36\pi \implies r^2 = 36 \implies r = 6$. Circumference $= 2\pi r = 2\pi(6) = 12\pi$.

3. **D** — Volume $= l \times w \times h = 5 \times 4 \times 3 = 60$ m^3. Converting to liters: $60 \times 1000 = 60,000$ liters.
4. **B** — The sum of exterior angles for *any* convex polygon is always exactly $360°$, regardless of the number of sides.

5 Statistics and Probability

5.1 Statistics

Definitions of Basic Terms

- **Statistics**
  Statistics is the science of collecting, organizing, presenting, analyzing and interpreting data in order to draw conclusions.

- **Descriptive Statistics**
  Descriptive statistics involves methods used to summarize and describe the main features of a dataset using tables, graphs, and summary measures such as mean and standard deviation.
  Examples: mean, median, mode, standard deviation, tables, graphs.

- **Inferential Statistics**
  Inferential statistics refers to techniques that use sample data to make estimates, predictions, or generalizations about a population.
  Examples: hypothesis testing, confidence intervals, regression analysis.

- **Population**
  A population is the complete set of individuals, items, or observations of interest in a study.
  Example: All malaria patients in Ethiopia.

- **Sample**
  A sample is a subset of the population selected for analysis.
  Example: 500 malaria patients selected from hospitals.

- **Parameter**
  A parameter is a numerical characteristic of a population, such as the population mean μ or population variance σ^2.

- **Statistic**
  A statistic is a numerical value calculated from a sample, such as the sample mean $\bar{x}$ or sample variance s^2.

- **Variable**
  A variable is a characteristic or attribute that can take different values.

- **Qualitative Variable**
  A non-numerical variable representing categories (e.g., gender, blood type).

- **Quantitative Variable**
  A numerical variable representing counts or measurements.

- **Discrete Variable**
  It takes countable values (e.g., number of patients).

- **Continuous Variable**
  It takes values within a continuous range (e.g., height, weight).

- **Data**
  Data are the observed values or measurements collected for analysis. And can be classified as primary data and secondary data.
  1. **Primary data**
  This is data collected first-hand by the researcher for a specific purpose.
     - ☞ Collected directly from the source.
     - ☞ Original and more reliable for the study at hand.
     - ☞ Methods of collections are surveys and questionnaires, interviews, experiments, direct observation etc.

  2. **Secondary data**
  This is data already collected and published by others for a different purpose.
     - ☞ Not collected by the current researcher.
     - ☞ Usually easier and cheaper to obtain.
     - ☞ Sources include books and journals, government reports (e.g., WHO, national statistics offices), websites and databases, previous research studies, etc.

- A **frequency distribution** is a tabular or graphical representation of data showing the frequency associated with each data value.

- A **histogram** is a graphical representation of a frequency distribution in which the variate (v) is plotted on the x-axis (horizontal axis) and the frequency (f) is plotted on the y-axis (vertical axis).

5.1.1 Measures of central tendency

A measure of **central tendency** is a value that represents the center point of a distribution of data.
Common Measures of Central Tendency are:

- ✍ Mean: The arithmetic average of all values. Affected by extreme values and unique for the given data.

- ✍ Median: The middle value when data are arranged in order. It is not affected by extreme values and unique for the given data.

- ✍ Mode: The most frequently occurring value. Mode is not unique (there can be more than one mode for a dataset). It is not affected by extreme values.

Measure of Central Tendency and Shape of Distribution

How the three measures of central tendency relate to one another depends entirely on the shape of the distribution:

- ◇ **Symmetrical (Normal) Distribution**
  The mean, median, and mode are all identical and located right in the center of the curve.
- ◇ **Positively Skewed (Skewed Right)**
  The distribution has a long tail dragging out to the right. Here, Mean > Median > Mode. The mean gets pulled toward the high-value tail.
- ◇ **Negatively Skewed (Skewed Left)**
  The distribution has a long tail dragging out to the left. Here, Mean < Median < Mode. The mean gets pulled toward the low-value tail.

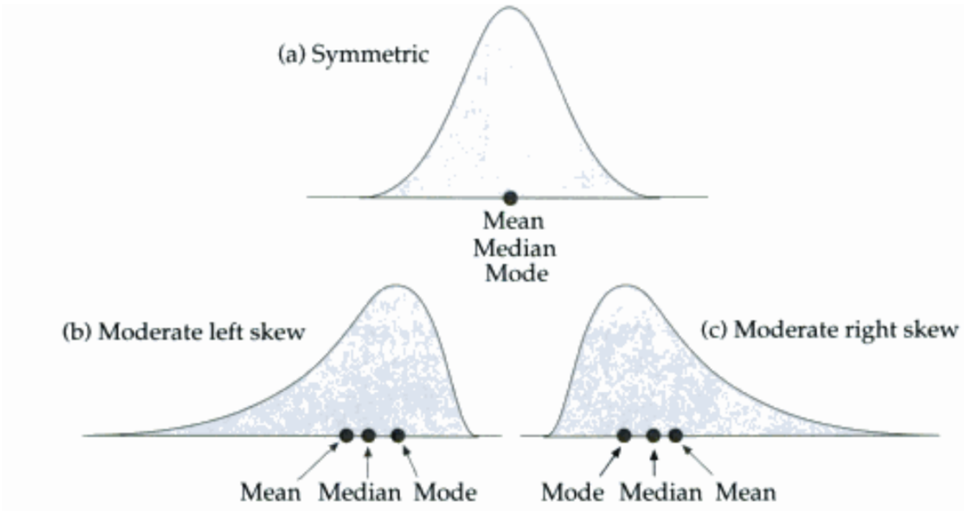


Figure 1: Distribution Shapes

For moderately skewed distributions:

- $\text{Mean} - \text{Mode} = 3(\text{Mean} - \text{Median})$ or equivalently: $\text{Mode} = 3\,\text{Median} - 2\,\text{Mean}$

5.1.2 Measures of dispersion

Measure of dispersion refers to statistical measures that describe how data values are *spread out* or *scattered* around a central value (such as the mean or median).
Common measures of dispersion are:

- ✍ Range
  The simplest measure of dispersion, calculated as the difference between the maximum and minimum values in a dataset.
  Moreover, it is very easy to calculate, but highly sensitive to outliers and ignores how the rest of the data is distributed between the extremes.

✍ Variance
Variance (σ^2) is defined as the mean of the squared deviations of each value from the arithmetic mean. It measures the average squared deviation from the mean and is sensitive to extreme values.

✍ Standard deviation
Standard deviation (σ) is defined as the positive square root of the mean of the squared deviations of each value from the arithmetic mean. It is the most commonly used measure of dispersion and is sensitive to extreme values.

Notes to Remember:

- If the data set entries are completely identical, all measures of dispersion are zero.
- When a constant c is added to every observation:
  - Mean shifts by c.
  - Variance and standard deviation remain entirely unchanged.
- If every value in a population is multiplied by a constant c:
  - The mean is multiplied by the constant c.
  - Variance is multiplied by the square of the constant (c^2).
  - Standard deviation is multiplied by the absolute value of the constant ($|c|$).
- Low dispersion means the population behaves uniformly (homogeneous).
- High dispersion suggests distinct subgroups or high volatility (heterogeneous).
- High standard deviation in a parameter (like transmission rate or drug clearance time) introduces significant structural uncertainty into calculations.

5.1.3 Inferential Statistics

Inferential statistics is a branch of statistics that uses sample data to make generalizations, predictions, or decisions about a population. Because it is often impractical to study an entire population, inferential methods allow researchers to draw reliable conclusions while accounting for uncertainty through probability.

1. **Estimation**
   Estimation involves using sample data to approximate unknown population parameters such as the mean (μ), proportion (p), or variance (σ^2).

   (a) **Point Estimation**
   A point estimate is a single numerical value used to estimate a population parameter.
   Example: Sample mean $\bar{x}$ estimates population mean μ. Sample proportion $\hat{p}$ estimates population proportion (p).

   Properties of a good estimator:

- **Unbiasedness:** The expected value of the estimator equals the true parameter.
- **Consistency:** The estimate becomes more accurate as the sample size increases.
- **Efficiency:** The estimator has the minimum variance among all unbiased estimators.

Example (Application): If the average infection rate in a sample is 12%, we use 12% as a point estimate of the population infection rate.

(b) **Interval Estimation**
An interval estimate provides a range of values within which the population parameter is likely to lie, along with a specified confidence level.
The confidence interval for a population mean (when σ is known) is given by:

$$\bar{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

Each parameter in the formula is defined as follows:

- $\bar{x}$ **(Sample Mean):** The average value computed from the sample data. It serves as the point estimate of the population mean μ.
- $z_{\alpha/2}$ **(Critical Value):** The value obtained from the standard normal distribution corresponding to the desired confidence level. Common values include:
  - 1.645 for a 90% confidence level
  - 1.96 for a 95% confidence level
  - 2.576 for a 99% confidence level
- σ **(Population Standard Deviation):** A measure of the variability or dispersion in the population. If σ is unknown, the sample standard deviation s is used instead with a t-distribution critical value if the sample size is small.
- n **(Sample Size):** The number of observations in the sample. A larger sample size reduces the width of the confidence interval, increasing precision.
- **Standard Error** ($\frac{\sigma}{\sqrt{n}}$)**:** Measures the variability of the sample mean from sample to sample, indicating the precision of the estimate.

Note:

- The confidence interval represents the range within which the true population parameter is expected to fall with a specified level of confidence.
- A 95% confidence interval means that if we repeat the sampling process indefinitely, approximately 95% of the calculated intervals will capture the true population parameter.
  Example: If a CI for an infection rate is $(10\%, 14\%)$, the true population infection rate is likely between 10% and 14%.

2. **Hypothesis Testing**
Hypothesis testing is a formal statistical procedure used to test claims or assumptions about a population parameter using sample data.

Null and Alternative Hypotheses

- **Null Hypothesis (H_0):** A statement of no effect, no difference, or the status quo.
  Example: "The treatment has no effect on transmission rates."
- **Alternative Hypothesis (H_1 or H_a):** A statement that contradicts the null hypothesis, representing the effect we hope to support.
  Example: "The treatment reduces the infection rate."

Types of Alternative Hypotheses:

- Two-tailed: $H_1 : \mu \neq \mu_0$
- Left-tailed: $H_1 : \mu < \mu_0$
- Right-tailed: $H_1 : \mu > \mu_0$

Types of Errors

When making statistical decisions, two types of errors can occur:

Type I Error (α)

Occurs when the null hypothesis H_0 is rejected when it is actually true (a **false positive**). It is controlled directly by setting the significance level α (e.g., $\alpha = 0.05$).
Example: Concluding that a drug is effective when it actually has no therapeutic value.

Type II Error (β)

Occurs when we fail to reject H_0 when it is actually false (a **false negative**). The probability of avoiding a Type II error $(1 - \beta)$ is known as the statistical power of a test.
Example: Concluding that a drug is ineffective when it actually works.

p-value

The p-value is the probability of obtaining test results at least as extreme as the results actually observed, under the assumption that the null hypothesis H_0 is true.

Decision Rule:

- If $p \leq \alpha \implies$ Reject H_0 (Statistically significant)
- If $p > \alpha \implies$ Fail to reject H_0 (Not statistically significant)

Interpretation:

- Small p-value: Strong sample evidence against H_0.
- Large p-value: Weak sample evidence against H_0.

Example:

- $p = 0.03, \alpha = 0.05 \implies$ Reject H_0
- $p = 0.10, \alpha = 0.05 \implies$ Fail to reject H_0

5.2 Probability

5.2.1 Introduction

Probability is a numerical value that describes how likely an event is to happen, ranging from 0 to 1.

- A probability of 0 means the event is impossible (e.g., rolling a 7 on a standard six-sided die).
- A probability of 1 means the event is certain (e.g., pulling a red ball out of a bag containing only red balls).

Elements of Probability

- An **Experiment** is any structured activity or process that generates observable results (outcomes).
- The **Sample Space** (or possibility set) S is the set of all possible outcomes of an experiment.
- An **Event** is any subset of outcomes contained within the sample space ($E \subseteq S$).

5.2.2 Probability Techniques

Probability techniques are mathematical methods used to evaluate uncertainty, randomness, and risk across fields like epidemiology, finance, and engineering.

1. Independence of Events

Two events A and B are independent if the occurrence of one does not alter the probability of the other:

$$P(A \cap B) = P(A)P(B)$$

2. Conditional Probability

The probability of an event A occurring given that event B has already occurred is defined as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad \text{where } P(B) > 0$$

3. Complement of an Event

The complement of an event A (denoted by A^c or A') represents the event that A does not occur:

$$P(A^c) = 1 - P(A)$$

4. Addition Rule

The probability that at least one of two events occurs is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive ($A \cap B = \emptyset$), the rule simplifies to:

$$P(A \cup B) = P(A) + P(B)$$

5. Multiplication Rule

The joint probability that two events occur together is:

$$P(A \cap B) = P(A)P(B \mid A) = P(B)P(A \mid B)$$

If independent, this simplifies directly to $P(A \cap B) = P(A)P(B)$.

5.2.3 Sampling Techniques

Sampling techniques designate the methodology used to select a representative subset of individuals from a population to infer characteristics of the population as a whole.

A. Probability Sampling

Probability sampling methods ensure every member of the population has a known, non-zero chance of being selected, minimizing selection bias.

1. **Simple Random Sampling:** Every individual has an equal chance of selection.
   Example: Assigning random numbers to 1,000 households and using a software generator to select 100 of them for a malaria screening study.
2. **Stratified Sampling:** The population is partitioned into distinct, homogeneous strata based on key traits (e.g., age, income), and random samples are collected proportionally from each.
   Example: Grouping a population into children, adults, and the elderly to ensure all age sub-cohorts are appropriately represented in a tuberculosis study.

3. **Cluster Sampling:** The population is partitioned into structural clusters (typically geographical). A random selection of clusters is chosen, and all or a sample of elements within those clusters are studied.
   Example: Selecting five districts at random within a region and surveying every household within those selected districts.

4. **Systematic Sampling:** Sampling elements are selected at fixed, regular intervals (every k-th element) from an ordered frame, starting from a randomly determined point.
   Example: Selecting every 10th patient who registers at a clinic for an HIV prevalence survey.

B. Non-Probability Sampling

In non-probability sampling, the selection probability for members of the population is unknown, often relying on convenience or human judgment.

1. **Quota Sampling:** The sample is compiled up to a fixed, non-random proportion of specified subgroups to mirror known population traits.
   Example: Recruiting participants on a street until exactly 50% males and 50% females are enrolled in a health-access questionnaire.

2. **Convenience Sampling:** Selecting cases based primarily on proximity, accessibility, and ease of recruitment.
   Example: Surveying fellow university classmates nearby to evaluate community awareness of disease prevention methods.

3. **Purposive (Judgmental) Sampling:** Participants are handpicked intentionally by the investigator based on professional expertise or specific structural requirements.
   Example: Restricting interview subjects exclusively to clinical experts who have managed multi-drug resistant tuberculosis (MDR-TB).

5.2.4 Errors in Sampling

Sampling deviations represent the discrepancy between a sample statistic and its true population parameter.

1. **Sampling Error:** The natural statistical variance that arises simply because a sample is observed instead of the entire population.
   - It is random and unavoidable.
   - Its magnitude decreases as the sample size n increases.

2. **Non-Sampling Errors:** Biases and operational flaws that arise during data collection, management, or design, impacting both samples and full censuses.

- **Response Error:** Respondents provide inaccurate, incomplete, or false metrics.
- **Measurement Error:** Faulty instrumentation or observational protocols distort data (e.g., miscalibrated diagnostic tools).
- **Non-Response Error:** Selected sample elements cannot be reached or refuse to participate.
- **Coverage Error:** An incomplete sampling frame systematically excludes segments of the target population.
- **Processing Error:** Structural slip-ups made during data entry, transcription, coding, or execution.
- **Interviewer Error:** Subtle or overt behavioral biases introduced by the investigator during data collection.

5.3 Regression Analysis

Regression analysis evaluates and models the functional relationship between a dependent outcome variable and one or more independent predictor variables.

5.3.1 Dependent Variable

The dependent variable (response or outcome variable) is the primary metric under investigation that we aim to model or forecast, denoted as Y.
Example: The daily active case count of an infectious disease outbreak.

5.3.2 Independent Variables

Independent variables (explanatory or predictor variables) are the underlying factors used to explain variations in the dependent variable, denoted as $X_1, X_2, \ldots, X_n$.
Example: Average ambient temperature, seasonal rainfall patterns, and vaccination coverage rates.

5.3.3 Regression Model

A multiple linear regression model is formally structured as:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_n X_n + \varepsilon$$

Where:

- Y is the dependent variable.
- X_i represents the independent predictor variables.

- β_0 is the y-intercept parameter.
- β_i designates the partial regression coefficients.
- ε represents the unobserved stochastic error term (assumed $\varepsilon \sim N(0, \sigma^2)$).

Interpretation of Coefficients

- $\beta_i > 0$: A positive relationship; as X_i increases, Y is expected to increase.
- $\beta_i < 0$: A negative relationship; as X_i increases, Y is expected to decrease.
- $\beta_i = 0$: No linear relationship exists between X_i and Y.

5.4 Correlation Analysis

Correlation analysis measures the strength and directional orientation of a linear association between two continuous variables. Crucially, correlation does not establish structural causation.

5.4.1 Types of Correlation

- **Positive Correlation:** Both variables move in tandem; as one increases, the other increases.
- **Negative Correlation:** Variables move inversely; as one increases, the other decreases.
- **Zero Correlation:** No discernable linear trend or association exists between the variables.

5.4.2 Correlation Coefficient

The Pearson product-moment correlation coefficient (r) is calculated as:

$$r = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2 \sum(y_i - \bar{y})^2}}$$

Where:

- x_i, y_i represent paired individual sample measurements.
- $\bar{x}, \bar{y}$ denote the respective sample means.

Interpretation of r:

- $r = +1$: Perfect positive linear correlation.
- $r = -1$: Perfect negative linear correlation.
- $r = 0$: No linear association present.
- $0 < |r| < 1$: Indicated degree of linear association (weak, moderate, or strong).

Important Formulas Summary

Mean:

$$\bar{x} = \frac{\sum x}{n}$$

Variance:

$$\sigma^2 = \frac{\sum(x - \bar{x})^2}{n}$$

Median:

Middle value of sorted data

Standard Deviation:

$$\sigma = \sqrt{\sigma^2}$$

Classical Probability:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes in } S}$$

Rules to Remember

- Always arrange raw data arrays in ascending or descending order before calculating the median.
- The standard deviation is fundamentally the positive square root of the variance, maintaining the original units of measurement.
- Valid probability values are strictly bounded by the interval: $0 \leq P(E) \leq 1$.

Short Notes

- **Mean:** The arithmetic average of a dataset.
- **Median:** The middle value dividing a sorted dataset into equal halves.
- **Mode:** The value that registers the highest frequency within a dataset.
- **Variance:** Measures the average squared deviations from the computed mean.

5.5 Examples

The following line graph gives the annual percent profit earned by a company during the period 1995–2000.

$$\% \text{ Profit} = \frac{\text{Income} - \text{Expenditure}}{\text{Expenditure}} \times 100$$

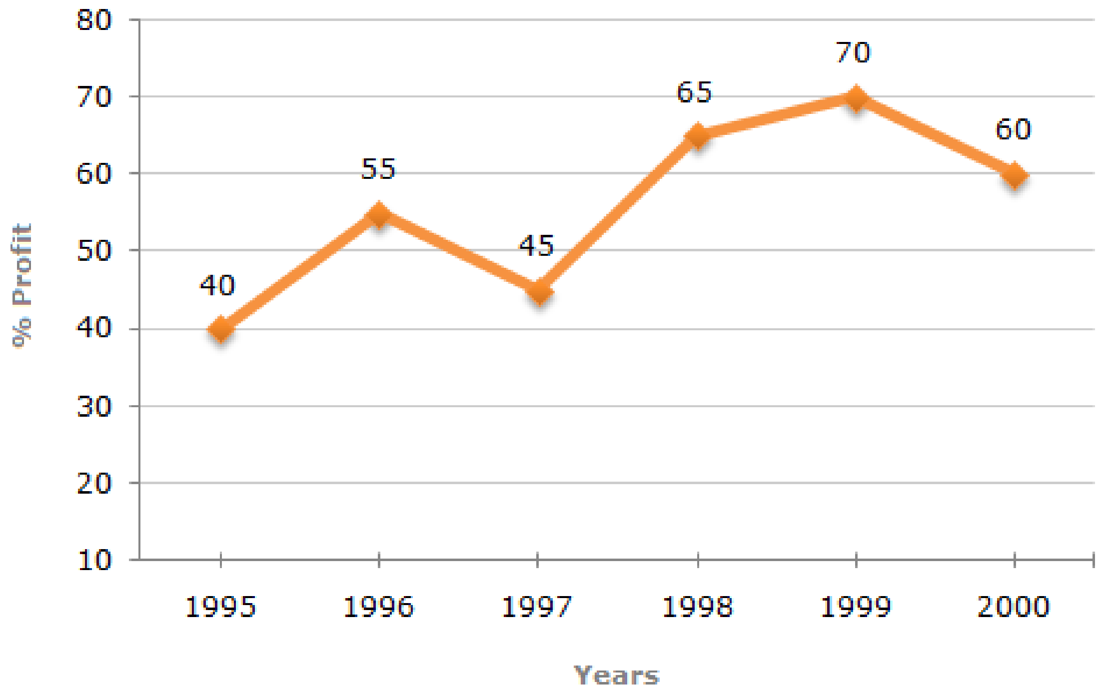


Figure 2: Line graph

1. If the expenditures in 1996 and 1999 are equal, then the approximate ratio of the income in 1996 and 1999 respectively is:

   (A) 1:1

   (B) 2:3

   (C) 13:14

   (D) 9:10

   Answer: D.
   Explanation: Let expenditure in 1996 = 1999 = x. Let incomes be I_1 and I_2.
   For 1996 (% Profit = 55):

   $$55 = \frac{I_1 - x}{x} \times 100 \implies I_1 = 1.55x$$

   For 1999 (% Profit = 70):

   $$70 = \frac{I_2 - x}{x} \times 100 \implies I_2 = 1.70x$$

   Ratio: $\frac{I_1}{I_2} = \frac{1.55x}{1.70x} = \frac{155}{170} \approx \frac{9}{10}$.

2. If the income in 1998 was 264 units, what was the expenditure in 1998?

   (A) 104 units
   (B) 145 units
   (C) 160 units
   (D) 185 units

   Answer: C.
   Explanation: Let expenditure in 1998 = x. The profit percentage in 1998 was 65%.

   $$65 = \frac{264 - x}{x} \times 100 \implies 0.65x = 264 - x \implies 1.65x = 264 \implies x = 160 \text{ units.}$$

3. In which year is the expenditure minimum?

   (A) 2000
   (B) 1997
   (C) 1996
   (D) Cannot be determined

   Answer: D.
   Explanation: The graph provides percentages only. Without baseline nominal values for income or expenditure in any given year, relative magnitudes cannot be calculated.

4. If the profit in 1999 was 4 units, what was the profit in 2000?

   (A) 4.2 units
   (B) 6.6 units
   (C) 6.8 units
   (D) Cannot be determined

   Answer: D.
   Explanation: Knowing absolute margins in 1999 tells us nothing about the scale of operation in 2000 without accompanying annual context.

5. What is the average profit earned for the given years?

   (A) $50\frac{2}{3}$
   (B) $55\frac{5}{6}$
   (C) $60\frac{1}{6}$
   (D) 335

   Answer: B.
   Explanation: Average $= \frac{40+55+45+65+70+60}{6} = \frac{335}{6} = 55\frac{5}{6}$.

6. For which of the following years, the percentage rise/fall in production from the previous year is the maximum for Company Y?

(A) 1997

(B) 1998

(C) 1999

(D) 2000

Answer: A.
Explanation: Calculating the sequential changes for Company Y based on data points:

- 1997: $\frac{35-25}{25} \times 100 = 40\%$
- 1998: $\frac{35-35}{35} \times 100 = 0\%$
- 1999: $\frac{40-35}{35} \times 100 = 14.29\%$
- 2000: $\frac{50-40}{40} \times 100 = 25\%$

7. What is the ratio of the average production of Company X (1998–2000) to Company Y (same period)?

(A) 1:1

(B) 15:17

(C) 23:25

(D) 27:29

Answer: C.
Explanation: $\text{Avg}_X = \frac{25+50+40}{3} = \frac{115}{3}$; $\text{Avg}_Y = \frac{35+40+50}{3} = \frac{125}{3}$. Ratio $= \frac{115}{125} = \frac{23}{25}$.

8. The average production for five years was maximum for which company?

(A) X

(B) Y

(C) Z

(D) X and Z both

Answer: D.
Explanation: $\text{Total}_X = 30+45+25+50+40 = 190$; $\text{Total}_Y = 25+35+35+40+50 = 185$; $\text{Total}_Z = 35 + 40 + 45 + 35 + 35 = 190$.

9. In which year was the percentage of production of Company Z to Company Y the maximum?

(A) 1996

(B) 1997

(C) 1998

(D) 1999

Answer: A.
Explanation: Evaluating proportions ($Z/Y \times 100$): 1996: $\frac{35}{25} = 140\%$; 1997: $\frac{40}{35} = 114.3\%$; 1998: $\frac{45}{35} = 128.6\%$; 1999: $\frac{35}{40} = 87.5\%$.

10. What is the percentage increase in the production of Company Y from 1996 to 1999?

(A) 30%
(B) 45%
(C) 50%
(D) 60%

Answer: D.
Explanation: $\frac{40-25}{25} \times 100 = \frac{15}{25} \times 100 = 60\%$.

11. A data set has a mean of 62. If each data point is decreased by a constant value of 7, what will the new mean be?

(A) 52
(B) 53
(C) 55
(D) 50

Answer: C.
Explanation: By linear transformation properties of estimators, $E[X - c] = E[X] - c$. Hence, $62 - 7 = 55$.

12. The mean salary paid to 10 employees of an establishment was found to be 200. Later on, it was discovered that the salary of one employee was wrongly entered as 180 instead of 240. What is the correct average salary?

(A) 194
(B) 200
(C) 206
(D) 210

Answer: C.
Explanation: $\bar{x}_{\text{correct}} = \bar{x}_{\text{wrong}} + \frac{x_{\text{actual}} - x_{\text{recorded}}}{n} = 200 + \frac{240-180}{10} = 206$.

13. Let A and B be two mutually exclusive events with $P(B) = 0.28$ and $P(A') = 0.47$. What is the probability of event A or B?

(A) 0.78
(B) 0.81
(C) 0.79
(D) 0.84

Answer: B.
Explanation: $P(A) = 1 - P(A') = 1 - 0.47 = 0.53$. Because they are mutually exclusive: $P(A \cup B) = P(A) + P(B) = 0.53 + 0.28 = 0.81$.

14. Tickets numbered 1 to 20 are mixed up and a ticket is drawn at random. What is the probability that the ticket drawn has a number which is a multiple of 3 or 5?

(A) $\frac{1}{2}$
(B) $\frac{2}{5}$
(C) $\frac{8}{15}$
(D) $\frac{9}{20}$

Answer: D.
Explanation: Sample space size $n(S) = 20$. Let E be the set of target multiples: $E = \{3, 6, 9, 12, 15, 18, 5, 10, 20\}$, meaning $n(E) = 9$. Thus, $P(E) = \frac{9}{20}$.

15. What is the probability of getting a sum of 9 from two throws of a die?

(A) $\frac{1}{6}$
(B) $\frac{1}{8}$
(C) $\frac{1}{9}$
(D) $\frac{1}{12}$

Answer: C.
Explanation: Total outcomes $n(S) = 36$. Acceptable outcomes: $E = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$ $n(E) = 4$. $P(E) = \frac{4}{36} = \frac{1}{9}$.

16. Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even?

(A) $\frac{1}{2}$
(B) $\frac{3}{4}$
(C) $\frac{3}{8}$
(D) $\frac{5}{16}$

Answer: B.
Explanation: Instead of calculating all elements explicitly, find the complementary probability of an odd product. A product is odd only if both dice show odd numbers: $3 \times 3 = 9$ paths. Thus, $P(\text{Even Product}) = 1 - \frac{9}{36} = \frac{27}{36} = \frac{3}{4}$.

17. Two dice are tossed. The probability that the total score is a prime number is:

(A) $\frac{1}{6}$
(B) $\frac{5}{12}$
(C) $\frac{1}{2}$
(D) $\frac{7}{9}$

Answer: B.
Explanation: Prime configurations for sums include: $2 \to (1, 1)$; $3 \to (1, 2), (2, 1)$; $5 \to (1, 4), (2, 3), (3, 2), (4, 1)$; $7 \to (1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$; $11 \to (5, 6), (6, 5)$. This yields $n(E) = 15 \implies P(E) = \frac{15}{36} = \frac{5}{12}$.

18. Suppose you asked ten classmates about their weight and found their average to state that the average weight of all students in your entire class is 60 kg. This is an example of:

(A) Descriptive Statistics
(B) Inferential Statistics
(C) Population
(D) None of the above

Answer: B.
Explanation: Extending sample results outwards to infer characteristics of an entire population represents a core objective of inferential methodology.

19. Which of the following is a qualitative variable:

(A) Ages of people living in a personal care home
(B) Number of students joining university every year
(C) Marital status of employees in a company
(D) Height of students measured in centimeters

Answer: C.
Explanation: Marital status yields definitive unranked categories, making it a qualitative/categorical variable.

20. Which of the following is an example of Quantitative Discrete data?

(A) Height of students in a class.
(B) The number of cars sold each month.
(C) Temperature in a city recorded hourly.
(D) The weight of a bag of sugar.

Answer: B.
Explanation: Counted metrics restricted strictly to whole integers describe discrete quantitative tracking.

21. Which of the following is classified as a ratio scale of measurement?

(A) Economic status classified as low, middle or upper
(B) Sex of person
(C) Eye color of individual
(D) Number of students in a class

Answer: D.
Explanation: Count data contains order, clear mathematical intervals, and a true zero baseline representing complete absence.

22. Classify the Blood type of individuals (A, B, AB, and O) as a type of measurement scale.

(A) Nominal
(B) Ordinal

(C) Interval
(D) Ratio

Answer: A.
Explanation: Blood groupings are purely categorical definitions with no mathematical priority or rank hierarchy.

23. Which of the following statements correctly describes the relationship between the variance (σ^2) and the standard deviation (σ)?

(A) The variance is always less than the standard deviation.
(B) The standard deviation is the square root of the variance.
(C) They are both measures of central tendency.
(D) Standard deviation is found by dividing the variance by the mean.

Answer: B.
Explanation: Standard deviation is definitionally isolated as the principal square root of computed variance.

24. Questionnaire method is used to collect:

(A) Primary data
(B) Secondary data
(C) A and B
(D) None of the above

Answer: A.
Explanation: Directly gathering data from respondents yields original, primary datasets.

25. Which of the following is different from the other:

(A) Quota sampling
(B) Convenience sampling
(C) Judgment sampling
(D) Systematic sampling

Answer: D.
Explanation: Systematic sampling uses random starting intervals, making it a probability approach; the others are non-probability strategies.

26. A researcher divided subjects into two groups according to gender and then selected members from each group for his/her sample. The sampling method used was:

(A) Cluster sampling
(B) Simple random sampling
(C) Stratified sampling
(D) None of the above

Answer: C.
Explanation: Grouping the broad dataset into distinct homogenous subgroups before selection outlines stratified sampling.

27. A 95% confidence level means:

 (A) Parameter is 95% correct
 (B) 95% of calculated intervals contain the true parameter
 (C) Sample mean is 95%
 (D) Error is 95%

 Answer: B.
 Explanation: Long-run probability guarantees that 95% of intervals generated will capture the underlying true parameter.

28. Null hypothesis is:

 (A) Always true
 (B) Statement to be tested
 (C) Alternative statement
 (D) Sample mean

 Answer: B.
 Explanation: The baseline structural claim undergoing active statistical verification.

29. Type I error occurs when:

 (A) Accepting a true null hypothesis
 (B) Rejecting a true null hypothesis
 (C) Rejecting a false null hypothesis
 (D) Accepting a false null hypothesis

 Answer: B.
 Explanation: Rejecting a correct default model represents a Type I error baseline (false positive).

30. The p-value is:

 (A) Probability that null hypothesis is true
 (B) Probability of observed result under the null hypothesis
 (C) Mean of sample
 (D) Variance

 Answer: B.
 Explanation: Quantifies sample extremity assuming H_0 holds true.

31. If p-value $\leq \alpha$:

(A) Accept null hypothesis
(B) Reject null hypothesis
(C) Ignore result
(D) Increase sample size

Answer: B.
Explanation: Low relative probability signals sample evidence that deviates far enough from assumptions to confidently drop H_0.

32. The significance level (α) is:

(A) Probability of Type I error
(B) Probability of Type II error
(C) Mean value
(D) Standard deviation

Answer: A.
Explanation: Identifies our maximum threshold for committing a Type I error.

33. What is meant by $p < 0.05$?

(A) That participants had an average score of 0.05.
(B) The probability of results occurring by chance alone is equal to or less than 0.05.
(C) There is a 0.05 chance that the results are wrong.
(D) The probability of result is equal to 0.05.

Answer: B.
Explanation: Reflects that sample variations match the null hypothesis with a probability below our chosen alpha threshold.

5.6 Self test exercise

The following line graph shows the annual percent profit earned by a company during the period 2001–2006.

$$\% \text{ Profit} = \frac{\text{Income} - \text{Expenditure}}{\text{Expenditure}} \times 100$$

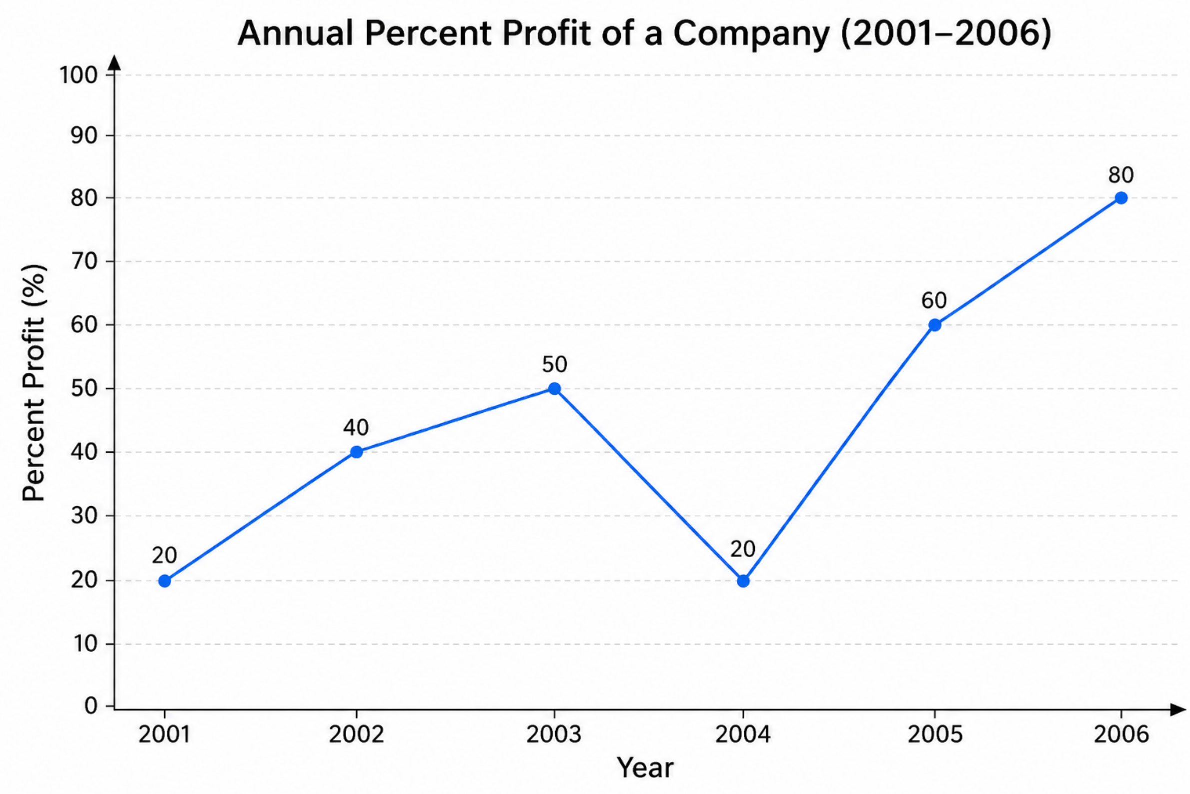


Figure 3: Line graph

1. If the expenditures in 2002 and 2005 are equal, then the approximate ratio of the income in 2002 and 2005 respectively is:

   (A) 1:1
   (B) 3:4
   (C) 7:8
   (D) 5:6

2. If the income in 2003 was 300 units and profit percentage was 50%, what was the expenditure?

   (A) 150 units
   (B) 180 units
   (C) 200 units
   (D) 225 units

3. In which year is the expenditure minimum, based only on profit percentages?

   (A) 2001
   (B) 2004
   (C) Cannot be determined
   (D) 2006

4. If the profit in 2004 was 6 units, what can be concluded about profit in 2005?

   (A) It must be higher
   (B) It must be lower
   (C) It cannot be determined
   (D) It is equal

5. The average profit percentage over 2001–2006 is given by:

   (A) Arithmetic mean of all yearly percentages
   (B) Weighted mean only
   (C) Geometric mean
   (D) Median only

6. If Company A shows production values: 40, 55, 50, 60, 65, find the average production.

   (A) 52
   (B) 54
   (C) 56
   (D) 58

7. Two dice are thrown. What is the probability of getting a sum of 8?

   (A) $\frac{5}{36}$
   (B) $\frac{1}{6}$
   (C) $\frac{7}{36}$
   (D) $\frac{1}{9}$

8. A card is drawn from numbers 1–30. What is the probability of a number divisible by 4 or 6?

   (A) $\frac{1}{3}$
   (B) $\frac{2}{5}$
   (C) $\frac{3}{10}$
   (D) $\frac{7}{30}$

9. The probability of drawing a prime number from 1–20 is:

   (A) $\frac{1}{2}$
   (B) $\frac{2}{5}$
   (C) $\frac{3}{10}$
   (D) $\frac{4}{10}$

10. If the mean of a dataset is 80 and each value is increased by 5, the new mean is:
    (A) 75
    (B) 80
    (C) 85
    (D) 90

11. The correct average salary of 8 employees is affected if one salary was recorded as 120 instead of 200. What happens to the mean?
    (A) It increases
    (B) It decreases
    (C) It remains unchanged
    (D) Cannot be determined

12. If $P(A) = 0.6$ and $P(B) = 0.3$ with mutually exclusive events, then $P(A \cup B)$ is:
    (A) 0.18
    (B) 0.90
    (C) 0.70
    (D) 0.60

13. A die is thrown. Probability of getting an even number or a number greater than 4 is:
    (A) $\frac{1}{2}$
    (B) $\frac{2}{3}$
    (C) $\frac{5}{6}$
    (D) $\frac{3}{4}$

14. Which of the following is a qualitative variable?
    (A) Age
    (B) Height
    (C) Gender
    (D) Weight

15. Which of the following is quantitative discrete data?
    (A) Temperature
    (B) Weight
    (C) Number of books
    (D) Height

16. Blood group classification belongs to:
    - (A) Ordinal scale
    - (B) Nominal scale
    - (C) Interval scale
    - (D) Ratio scale
17. The standard deviation is:
    - (A) Square of variance
    - (B) Square root of variance
    - (C) Always larger than variance
    - (D) Equal to mean
18. Questionnaire method is used to collect:
    - (A) Primary data
    - (B) Secondary data
    - (C) Both
    - (D) None
19. A sample is divided into groups based on income level before sampling. This method is:
    - (A) Cluster sampling
    - (B) Stratified sampling
    - (C) Systematic sampling
    - (D) Random sampling
20. If p-value ≤ 0.05, we:
    - (A) Accept H_0
    - (B) Reject H_0
    - (C) Increase sample size
    - (D) Ignore result

Answers and Brief Explanations

1. **Answer: Depends on graph values (A–D) Explanation:** From the graph, compute income using:

   $$\text{Income} = \text{Expenditure}\left(1 + \frac{P}{100}\right)$$

   Then form the ratio for 2002 and 2005.

2. **Answer: (C) 200 units Explanation:**

$$300 = E(1 + 0.5) = 1.5E \Rightarrow E = 200$$

3. **Answer: (C) Cannot be determined Explanation:** Expenditure depends on both income and profit

4. **Answer: (C) It cannot be determined Explanation:** Profit value alone does not determine trend without expenditure or percentage.

5. **Answer: (A) Arithmetic mean of all yearly percentages Explanation:** Each year is equally weighted in the time period.

6. **Answer: (B) 54 Explanation:**

$$\frac{40 + 55 + 50 + 60 + 65}{5} = \frac{270}{5} = 54$$

7. **Answer: (A) $\frac{5}{36}$ Explanation:** Pairs giving sum 8: (2,6), (3,5), (4,4), (5,3), (6,2) $\rightarrow$ 5 outcomes out of 36.

8. **Answer: (D) $\frac{7}{30}$ Explanation:** Multiples of 4: 7 numbers, multiples of 6: 5 numbers, overlap (12): 2 numbers.

$$7 + 5 - 2 = 10 \Rightarrow \frac{10}{30} = \frac{1}{3}$$

   Correct simplified result: $\frac{1}{3}$.

9. **Answer: (B) $\frac{2}{5}$ Explanation:** Primes in 1–20: 2,3,5,7,11,13,17,19 $\rightarrow$ 8 numbers:

$$\frac{8}{20} = \frac{2}{5}$$

10. **Answer: (C) 85 Explanation:** Mean increases by the same constant:

$$80 + 5 = 85$$

11. **Answer: (B) It decreases Explanation:** Recorded value is lower than actual (120 instead of 200), so mean decreases.

12. **Answer: (B) 0.90 Explanation:**

$$P(A \cup B) = P(A) + P(B) = 0.6 + 0.3 = 0.9$$

13. **Answer: (C) $\frac{2}{3}$ Explanation:** Even numbers: 2,4,6; greater than 4: 5,6 Union = 2,4,5,6 $\rightarrow$ 4 outcomes:

$$\frac{4}{6} = \frac{2}{3}$$

14. **Answer: (C) Gender Explanation:** It is categorical, not numerical.

15. **Answer: (C) Number of books Explanation:** It is countable discrete data.

16. **Answer: (B) Nominal scale Explanation:** Blood groups are labels without order.

17. **Answer: (B) Square root of variance Explanation:** Standard deviation = $\sqrt{\text{variance}}$.

18. **Answer: (A) Primary data Explanation:** Questionnaires collect data directly from respondents.

19. **Answer: (B) Stratified sampling Explanation:** Population is divided into groups before sampling.

20. **Answer: (B) Reject H_0 Explanation:** $p \leq 0.05$ indicates statistical significance.

6 Word Problems with One and Two Variables

Short Notes

A word problem is a mathematical challenge presented as a narrative scenario describing a real-world system or relationship rather than an explicit algebraic equation. To unpack these, you must extract key parameters, define variables, and set up deterministic systems of algebraic expressions.

1. **Single-Variable Word Problems:** Focus on a single unknown factor or map all secondary variables to expressions dependent entirely on that primary variable.
2. **Two-Variable Word Problems:** Involve two distinct unknowns, which typically require a system of independent simultaneous equations to reach a unique solution.

Short Notes

A word problem is a mathematical problem presented in the form of a real-life situation or written description rather than as a direct equation. To solve it, you must interpret the information given in words, identify the unknown quantities, translate the statements into mathematical expressions or equations, and then perform the necessary calculations.

1. **Interpret the information given in words carefully** Read the problem slowly and understand the situation described. Pay attention to what is being said and what is being asked before doing any calculations.
2. **Identify the unknown quantities** Determine what you need to find. These are usually the missing values in the problem, such as a number, age, length, or quantity.
3. **Translate the statements into mathematical expressions or equations** Convert the written information into symbols and equations. For example, phrases like "sum of two numbers" become $x + y$, and "is equal to" becomes =.
4. **Perform the necessary calculations to find the solution** Solve the equations using appropriate methods such as substitution, elimination, or simple arithmetic, then obtain the final answer.

Examples of Translating Words into Equations:

- "The sum of two numbers is 20" $\rightarrow x + y = 20$
- "One number exceeds another by 6" $\rightarrow x = y + 6$ or $x - y = 6$
- "Twice a number is 18" $\rightarrow 2x = 18$

- “A number decreased by 5 is 12” $\to x - 5 = 12$
- “The product of two numbers is 24” $\to xy = 24$
- “One number is three times another” $\to x = 3y$
- “Half of a number is 10” $\to \frac{x}{2} = 10$ or $x = 20$
- “The difference between two numbers is 8” $\to x - y = 8$ or $y = x - 8$
- “A number is 7 more than another” $\to x = y + 7$

Important Formulas

Core equations used in word problems:

- Sum of two numbers: $x + y = S$
- Difference of two numbers: $x - y = D$
- Product of two numbers: $xy = P$
- Quotient of two numbers: $\frac{x}{y} = Q$
- One number in terms of another (greater than): $x = y + a$
- One number in terms of another (less than): $x = y - a$
- Percentage form: $\text{Part} = \frac{\text{Percentage}}{100} \times \text{Whole}$
- Distance formula (motion problems): $D = vt$
- Age relations: $\text{Present age} = \text{Past age} + t$
- Sum and difference identities: $x = \frac{S+D}{2}, \quad y = \frac{S-D}{2}$

Rules to Remember

- Define variables explicitly before writing out systems of equations.
- Read every limiting condition carefully to check for unstated domain boundaries (e.g., ages must be positive integers).
- Use algebraic substitution or elimination systematically for systems of linear equations.
- Always check answers by substituting them back into the original textual constraints rather than your formulated equations.
- NGAT questions often hide crucial mathematical constraints in structural keywords such as *sum, difference, greater than, less than, twice,* and *half.*

6.1 Examples

Solved Problem 1: Single-Variable Linear Formulation

Problem. When a certain number is decreased by 15, the result is equal to three times the number minus 65. Find the number.

Solution:
Let the unknown number be x.

- "A number decreased by 15" translates to: $x - 15$
- "Three times the number minus 65" translates to: $3x - 65$

Set up the linear equation based on the condition of equality:

$$x - 15 = 3x - 65$$

Isolate the variable x by subtracting x from both sides:

$$-15 = 2x - 65$$

Add 65 to both sides:

$$50 = 2x$$

Divide by 2:

$$x = 25$$

Answer:
$\boxed{x = 25}$

Solved Problem 2: Two-Variable Sum and Difference System

Problem. The sum of two numbers is 84 and their difference is 22. Find both numbers.

Solution:
Let the larger number be x and the smaller number be y. We are given:

$$x + y = 84 \quad (\text{Sum } S)$$

$$x - y = 22 \quad (\text{Difference } D)$$

Using the sum-and-difference identities provided in the core formula guide:

$$x = \frac{S + D}{2} = \frac{84 + 22}{2} = \frac{106}{2} = 53$$

$$y = \frac{S - D}{2} = \frac{84 - 22}{2} = \frac{62}{2} = 31$$

Answer:
$\boxed{x = 53, \qquad y = 31}$

Solved Problem 3: Age Relations (Temporal Translation)

Problem. A mother is currently three times as old as her daughter. In 12 years, the mother will be exactly twice as old as her daughter will be then. Find their present ages.

Solution:
Let the present age of the daughter be d. Consequently, the present age of the mother is $3d$.

- Daughter's age in 12 years: $d + 12$
- Mother's age in 12 years: $3d + 12$

Set up the relationship for 12 years into the future:

$$\text{Mother's future age} = 2 \times (\text{Daughter's future age})$$

$$3d + 12 = 2(d + 12)$$

Expand the right side of the linear relation:

$$3d + 12 = 2d + 24$$

Subtract $2d$ from both sides:

$$d + 12 = 24 \implies d = 12$$

Substitute back to find the mother's present age: $3d = 3(12) = 36$.
Answer:
$\boxed{\text{Mother's age} = 36 \text{ years}, \quad \text{Daughter's age} = 12 \text{ years}}$

Solved Problem 4: Multi-Variable Value Systems

Problem. A candidate buys a total of 30 academic notebooks for an outreach program. Standard lined notebooks cost \$3 each, while specialized graph notebooks cost \$5 each. If the total expenditure is \$114, how many graph notebooks did they purchase?

Solution:
Let x be the number of standard lined notebooks, and y be the number of graph notebooks.

$$x + y = 30 \quad \text{(Quantity Equation)}$$

$$3x + 5y = 114 \quad \text{(Cost Equation)}$$

Isolate x from the quantity equation: $x = 30 - y$. Substitute this into the cost equation:

$$3(30 - y) + 5y = 114$$

$$90 - 3y + 5y = 114$$

Combine like terms:

$$90 + 2y = 114$$

$$2y = 24 \implies y = 12$$

Thus, the number of graph notebooks purchased is 12.
Answer:
$\boxed{\text{12 graph notebooks}}$

Solved Problem 5: Uniform Motion (D

Problem. Two courier vans leave the same logistics hub at the same time, traveling in opposite directions. One van travels at an average speed of 65 km/h, and the other travels at 85 km/h. After how many hours will they be 450 km apart?

Solution:
Let t represent the total transit time in hours.

- Distance covered by Van 1: $D_1 = 65t$
- Distance covered by Van 2: $D_2 = 85t$

Since they are moving in opposite directions away from a shared starting node, the total separation distance is the sum of their individual distances:

$$D_1 + D_2 = 450$$

$$65t + 85t = 450$$

Combine coefficients:

$$150t = 450$$

Divide by 150:

$$t = 3$$

Answer:
$\boxed{\text{3 hours}}$

Solved Problem 6: Geometric Dimension Constraints

Problem. The perimeter of an examination hall is 72 meters. The length of the hall is 6 meters less than twice its width. Find the exact dimensions of the hall.

Solution:
Let W be the width of the hall. The length L is expressed as: $L = 2W - 6$.
The formula for the perimeter of a rectangle is:

$$P = 2L + 2W$$

Substitute the perimeter total (72) and our length expression into the identity:

$$72 = 2(2W - 6) + 2W$$

Expand and consolidate:

$$72 = 4W - 12 + 2W$$

$$72 = 6W - 12$$

Add 12 to both sides:

$$84 = 6W \implies W = 14 \text{ meters}$$

Calculate length: $L = 2(14) - 6 = 28 - 6 = 22$ meters.
Answer:
$\boxed{\text{Length} = 22 \text{ m}, \quad \text{Width} = 14 \text{ m}}$

Solved Problem 7: Fractional Values and Ratio Logic

Problem. One number is $\frac{2}{3}$ of another number. If their sum is 75, what is the value of the larger number?

Solution:
Let the larger number be x. The smaller number is therefore $\frac{2}{3}x$.
The problem states that their sum equals 75:

$$x + \frac{2}{3}x = 75$$

Find a common denominator to combine the terms on the left side:

$$\frac{3}{3}x + \frac{2}{3}x = 75 \implies \frac{5}{3}x = 75$$

Multiply both sides by 3 to eliminate the fraction:

$$5x = 225$$

Divide by 5 to isolate x:

$$x = 45$$

The smaller number would be $\frac{2}{3}(45) = 30$, meaning 45 is indeed the larger number.
Answer:
$\boxed{45}$

Solved Problem 8: Simultaneous Systems with Scaled Coefficients

Problem. Double a first number plus three times a second number equals 38. Five times the first number minus twice the second number equals 19. Find the two numbers.

Solution:
Let the first number be x and the second number be y. Formulate the linear system:

$$2x + 3y = 38 \quad \text{— (Equation 1)}$$

$$5x - 2y = 19 \quad \text{— (Equation 2)}$$

Use the method of elimination. Multiply Equation 1 by 2 and Equation 2 by 3 to eliminate y:

$$4x + 6y = 76$$

$$15x - 6y = 57$$

Add the two modified equations together:

$$(4x + 15x) + (6y - 6y) = 76 + 57$$

$$19x = 133 \implies x = 7$$

Substitute $x = 7$ back into Equation 1 to find y:

$$2(7) + 3y = 38 \implies 14 + 3y = 38 \implies 3y = 24 \implies y = 8$$

Answer:
$\boxed{x = 7, \qquad y = 8}$

Solved Problem 9: Percentage Form Word Problems

Problem. An administrator allocates a research grant budget such that 40% is spent on lab equipment, and the remaining $18,000 is used for field operational costs. What was the total value of the research grant?

Solution:
Let T be the total value of the research grant.

- Percentage spent on lab equipment: 40%
- Percentage remaining for operations: $100\% - 40\% = 60\%$

Using the structural percentage identity (Part $= \frac{\text{Percentage}}{100} \times$ Whole):

$$18,000 = \frac{60}{100} \times T$$

Simplify the fractional multiplier:

$$18,000 = 0.6T$$

Divide both sides by 0.6 to isolate T:

$$T = \frac{18,000}{0.6} = 30,000$$

Answer:
$\boxed{\$30,000}$

Solved Problem 10: Advanced Ratio-Shift Framework

Problem. The current ratio of undergraduate assistants to graduate assistants in a department is 5 : 2. If the department hires 6 more graduate assistants, the new ratio becomes 5 : 3. How many undergraduate assistants are currently in the department?

Solution:
Let the original number of undergraduate assistants be $5k$ and graduate assistants be $2k$, where k is a common scalar factor.
The number of graduate assistants increases by 6, making the new total $2k + 6$.
Set up the new ratio expression:

$$\frac{5k}{2k+6} = \frac{5}{3}$$

Cross-multiply to solve for k:

$$3(5k) = 5(2k+6)$$

$$15k = 10k + 30$$

Subtract $10k$ from both sides:

$$5k = 30 \implies k = 6$$

The current number of undergraduate assistants is defined as $5k$:

$$5(6) = 30$$

Answer:
$\boxed{\text{30 undergraduate assistants}}$

6.2 Self test exercise

Attempt all questions without looking at the solutions.

1. The sum of two numbers is 24 and their difference is 8. What is the larger number?
   (A) 12
   (B) 14
   (C) 16
   (D) 18
2. A number increased by 7 equals 22. What is the number?
   (A) 13
   (B) 15
   (C) 17

(D) 29

3. The sum of two numbers is 40. One number is 12 greater than the other. Find the smaller number.

   (A) 14
   (B) 16
   (C) 18
   (D) 20

4. Twice a number plus 5 equals 25. Find the number.

   (A) 8
   (B) 10
   (C) 12
   (D) 15

5. A father is 30 years older than his son. Their ages add up to 54 years. How old is the son?

   (A) 10
   (B) 12
   (C) 15
   (D) 18

6. The sum of two numbers is 50 and their difference is 14. Find the numbers.

   (A) 32 and 18
   (B) 36 and 14
   (C) 30 and 20
   (D) 34 and 16

7. Five times a number minus 7 equals 28. Find the number.

   (A) 5
   (B) 6
   (C) 7
   (D) 8

8. The perimeter of a rectangle is 36 cm. The length is 4 cm greater than the width. Find the dimensions.

   (A) Length = 10 cm, Width = 8 cm
   (B) Length = 12 cm, Width = 6 cm
   (C) Length = 11 cm, Width = 7 cm
   (D) Length = 9 cm, Width = 5 cm

9. A mother is 24 years older than her daughter. Their ages sum to 40 years. Find their ages.
   (A) Mother = 30 years, Daughter = 10 years
   (B) Mother = 32 years, Daughter = 8 years
   (C) Mother = 28 years, Daughter = 12 years
   (D) Mother = 26 years, Daughter = 14 years
10. The sum of two numbers is 60. One number is twice the other. Find the numbers.
   (A) 20 and 40
   (B) 15 and 45
   (C) 18 and 42
   (D) 25 and 35

Challenge Question

A basket contains chickens and rabbits. There are 20 heads and 56 legs altogether. How many chickens and rabbits are there in the basket?

Answer Key & Detailed Explanations

1. **Answer: C**
   Let numbers be x and y. Using formulas: $x = \frac{S+D}{2} = \frac{24+8}{2} = 16$.

2. **Answer: B**
   Equation: $x + 7 = 22 \implies x = 22 - 7 = 15$.

3. **Answer: A**
   Let smaller be y, larger be $y + 12$. System: $y + (y + 12) = 40 \implies 2y = 28 \implies y = 14$.

4. **Answer: B**
   Equation: $2x + 5 = 25 \implies 2x = 20 \implies x = 10$.

5. **Answer: B**
   Let son's age be s, father's be $s+30$. System: $s+(s+30) = 54 \implies 2s = 24 \implies s = 12$.

6. **Answer: D**
   Larger $x = \frac{50+14}{2} = 32$; Smaller $y = \frac{50-14}{2} = 18$.

7. **Answer: C**
   Equation: $5x - 7 = 28 \implies 5x = 35 \implies x = 7$.

8. **Answer: C**
   Perimeter $2(L+W) = 36 \implies L+W = 18$. Given $L = W+4 \implies (W+4)+W = 18 \implies 2W = 14 \implies W = 7$ cm and $L = 11$ cm.

9. **Answer: B**
   Mother $M = D + 24$. Sum: $(D + 24) + D = 40 \implies 2D = 16 \implies D = 8$, $M = 32$.

10. **Answer: A**
    Equations: $x + y = 60$ and $x = 2y \implies 2y + y = 60 \implies 3y = 60 \implies y = 20, x = 40$.

11. **Challenge Solution:**
    Let chickens be c and rabbits be r.

    $$c + r = 20 \quad \text{(Heads Equation)}$$

    $$2c + 4r = 56 \quad \text{(Legs Equation)}$$

    Multiply the heads equation by 2: $2c+2r = 40$. Subtract this parameter map from the legs equation:

    $$(2c + 4r) - (2c + 2r) = 56 - 40 \implies 2r = 16 \implies r = 8$$

    Substitute back to find chickens: $c = 20 - 8 = 12$.
    Final Answer: 12 Chickens and 8 Rabbits.

Exam Tip

(a) When you read carefully, you start to understand how the quantities are connected. For example, you can see which value depends on another, or how one value changes the other. That relationship is what you translate into mathematics.

(b) Most mistakes happen when students jump straight into solving without understanding these structural connections. They may choose the wrong variable dependencies, set up the wrong linear system, or miss an important boundary condition completely.

(c) So the key idea is simple: slow down, understand the narrative layout first, then construct your mathematical models. Once the structural relationships are clear, completing the algebraic manipulation becomes much easier and far less prone to errors.